This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Daw[illegible]

3-0[illegible]

ELEMENTS OF ALGEBRA:

ON

THE BASIS OF M. BOURDON:

EMBRACING

STURM'S AND HORNER'S THEOREMS.

REVISED AND RE-WRITTEN

By CHARLES DAVIES, LL.D.,

AUTHOR OF ARITHMETIC, ELEMENTARY ALGEBRA, ELEMENTARY GEOMETRY, PRACTICAL GEOMETRY, ELEMENTS OF SURVEYING, ELEMENTS OF DESCRIPTIVE AND ANALYTICAL GEOMETRY, ELEMENTS OF DIFFERENTIAL AND INTEGRAL CALCULUS, AND A TREATISE ON SHADES, SHADOWS, AND PERSPECTIVE.

A. S. BARNES AND COMPANY,
NEW YORK AND CHICAGO.
1875.

Ia
3-C

THE NEW YORK
PUBLIC LIBRARY
828224
ASTOR, LENOX AND
TILDEN FOUNDATIONS
1919 L

DAVIES' COURSE OF MATHEMATICS.

IN THREE PARTS.

I. COMMON SCHOOL COURSE.

Davies' Primary Arithmetic.—The fundamental principles displayed in Object Lessons.

Davies' Intellectual Arithmetic.—Referring all operations to the unit 1 as the only tangible basis for logical development.

Davies' Elements of Written Arithmetic.—A practical introduction to the whole subject. Theory subordinated to Practice.

Davies' Practical Arithmetic.—A combination of Theory and Practice, clear, exact, brief, and comprehensive.

II. ACADEMIC COURSE.

Davies' University Arithmetic.—Treating the subject exhaustively as a *science*, in a logical series of connected propositions.

Davies' Elementary Algebra.—A connecting link, conducting the pupil easily from arithmetical processes to abstract analysis.

Davies' University Algebra.—For institutions desiring a more complete but not the fullest course in pure Algebra.

Davies' Practical Mathematics.—The science practically applied to the useful arts, as Drawing, Architecture, Surveying, Mechanics, etc.

Davies' Elementary Geometry.—The important principles in simple form, but with all the exactness of rigorous reasoning.

Davies' Elements of Surveying.—Re-written in 1870. A simple and full presentation for Instruction and Practice.

III. COLLEGIATE COURSE.

Davies' Bourdon's Algebra.—Embracing Sturm's Theorem, and a most exhaustive course. Re-written, in 1873.

Davies' University Algebra.—A shorter course than Bourdon, for Institutions having less time to give the subject.

Davies' Legendre's Geometry.—A standard work in this country and in Europe.

Davies' Analytical Geometry.—A full course of Analysis, embracing the applications to surfaces of the second order.

Davies' Differential and Integral Calculus, on the basis of Continuous Quantity and Consecutive Differences.

Davies' Analytical Geometry and Calculus.—The shorter treatises, combined in one volume.

Davies' Descriptive Geometry.—With application to Spherical Trigonometry, Spherical Projections, and Warped Surfaces.

Davies' Shades, Shadows, and Perspective.—A succinct exposition of the mathematical principles involved.

Davies' Nature and Utility of Mathematics, Logically considered.

Davies and Peck's Mathematical Dictionary, or Cyclopedia of Mathematics.

Entered according to Act of Congress, in the year 1873, by
CHARLES DAVIES,
In the Office of the Librarian of Congress, at Washington.

PREFACE.

THE Treatise on Algebra, by M. Bourdon, is a work of singular excellence and merit. In France, it has long been one of the standard Text-Books. Shortly after its publication, it passed through several editions, and has since formed the basis of every work on the subject of Algebra, both in Europe and in this country.

In the present work, the original Treatise of M. Bourdon has been regarded only as a model. The order of arrangement, in many parts, has been changed; new rules and new methods have been introduced; the modifications indicated by its use, for thirty years, as a Text-Book in the Military Academy, have been freely made, for the purpose of giving to the work a more practical character, and bringing it into closer harmony with the trains of thought and improved systems of instruction which prevail in that institution.

A casual acknowledgment would fail to recognize my obligations to Professor Peck, of Columbia College, for the valuable labors he has bestowed on the present edition. Many of the new definitions, new rules, and improved methods of illustration, are his. His scientific attainments, long known to the public, and his experience as a teacher of Mathematics, impart to his writings the marks of profound study and the freshness of daily instruction.

The manner of treating the Binomial Formula, and the chapter on the General Theory of Equations, will, it is believed, be found worthy of special attention.

To lessen the labor of Teachers, a Key has been prepared, exhibiting in detail, all the difficult operations. To this Key, many examples and discussions, not contained in the Text, have been added, which will be found useful in classes as test questions, and as presenting new subjects of investigation.

FISHKILL-ON-HUDSON, }
June, 1873. }

CONTENTS.

PAGES
INTRODUCTION.. ix—xv

CHAPTER I.

DEFINITIONS AND INTRODUCTORY REMARKS.

ARTICLES
QUANTITY DEFINED.. 1
MATHEMATICS DEFINED 2
ALGEBRA 3
KINDS OF QUANTITY.. 4
SIGNS—OF OPERATION, OF RELATION, OF ABBREVIATION........ 5—11
COEFFICIENTS.. 11
TERMS.. 12
MONOMIALS, POLYNOMIALS, RECIPROCALS........................ 13—16
NUMERICAL VALUES.. 16

CHAPTER II.

FUNDAMENTAL OPERATIONS..

I. ADDITION.. 17
SIMILAR QUANTITIES. RULE.. 18—19
DISSIMILAR QUANTITIES. RULE.. 20
II. SUBTRACTION .... 21—22
BRACKETS—ESSENTIAL SIGN.. 23—24
III. MULTIPLICATION.. 25—31
RESULTS—USEFUL FORMULAS .... 31—34
IV. DIVISION.. 34—42
PRINCIPLES AND RESULTS.. 42—43
V. FACTORING .... 43—52
VI. GREATEST COMMON DIVISOR.. 52—56
VII. LEAST COMMON MULTIPLE.. 56—61

CHAPTER III.

ALGEBRAIC FRACTIONS.

ARTICLES
I. DEFINITIONS AND FIRST PRINCIPLES.................. 61—70
II. TRANSFORMATION OF FRACTIONS.................... 70—76
III. ADDITION OF FRACTIONS......................... 76—77
IV. SUBTRACTION OF FRACTIONS...................... 77—78
V. MULTIPLICATION OF FRACTIONS.................... 78—79
VI. DIVISION OF FRACTIONS......................... 79—80
VII. OF THE SYMBOLS 0, ∞, AND $\frac{0}{0}$ .... 80—81

CHAPTER IV.

EQUATIONS OF THE FIRST DEGREE.

I. DEFINITIONS AND FIRST PRINCIPLES.................. 81—87
II. SOLUTION OF EQUATIONS......................... 87—91
III. SOLUTION OF PROBLEMS......................... 91—92
IV. SIMULTANEOUS EQUATIONS....................... 92—93
V. ELIMINATION.................................... 93—98
VI. PRACTICAL PROBLEMS............................ 97—98
VII. INDETERMINATE EQUATIONS...................... 98—99
VIII. INTERPRETATION OF NEGATIVE RESULTS.......... 99—100
IX. DISCUSSION OF PROBLEMS........................ 100—103
X. OF INEQUALITIES................................ 103—104

CHAPTER V.

SQUARE ROOT AND RADICALS OF THE SECOND DEGREE.

I. SQUARE ROOT OF MEMBERS......................... 104—109
II. SQUARE ROOT OF ALGEBRAIC QUANTITIES........... 109—110
III. RADICALS OF THE SECOND DEGREE................ 110—119

CHAPTER VI.

EQUATIONS OF THE SECOND DEGREE.

I. EQUATIONS OF ONE UNKNOWN QUANTITY.............. 119—124
II. PROBLEMS IN QUADRATIC EQUATIONS............... 123—124

ARTICLES
III. PROPERTIES OF EQUATION OF SECOND DEGREE........ 124—132
IV. TRINOMIAL EQUATIONS .... 132—134
V. SOLUTION OF SIMULTANEOUS EQUATIONS.............. 134—138
VI. PRACTICAL PROBLEMS .... 137—138

CHAPTER VII.

FORMATION OF POWERS AND EXTRACTION OF ROOTS.

I. POWERS OF MONOMIALS............................ 138—141
II. BINOMIAL FORMULA .... 141—143
III. EXTRACTION OF ROOTS OF NUMBERS .... 143—149
IV. EXTRACTION OF ROOTS OF ALGEBRAIC QUANTITIES .... 149—151
V. TRANSFORMATION OF RADICALS .... 151—164
VI. FRACTIONAL EXPONENTS .... 164—168

CHAPTER VIII.

FORMATION, PROPERTIES, AND SUMMATION OF SERIES.

I. ARITHMETICAL PROGRESSION......................... 168—180
II. OF RATIO AND PROPORTION.......................... 180—185
III. GEOMETRICAL PROGRESSION.......................... 185—192
IV. IDENTICAL EQUATIONS.............................. 192—197
V. INDETERMINATE COEFFICIENTS........................ 197—199
RECURRING SERIES.................................. 199—201
VI. BINOMIAL FORMULA FOR ANY EXPONENT................ 201—205
SERIES .... 205—208
VII. SUMMATION OF SERIES.............................. 208—215

CHAPTER IX.

LOGARITHMS, INTEREST, COMBINATIONS, AND PROBABILITIES.

I. OF LOGARITHMS.................................... 215—234
II. CONTINUED FRACTIONS............................. 234—242
III. SIMPLE AND COMPOUND INTEREST.................. ... 242—247
IV. PERMUTATIONS, ARRANGEMENTS, AND COMBINATIONS ... 247—251
V. PROBABILITIES .... 251—255

CHAPTER X.

GENERAL THEORY OF EQUATIONS.

ARTICLES
I. PROPERTIES OF EQUATIONS .... 255—259
COMPOSITION OF EQUATIONS .... 259—260
II. TRANSFORMATION OF EQUATIONS .... 260—265
SYNTHETICAL DIVISION .... 265—266
DISAPPEARANCE OF SECOND TERM .... 266—267
III. DERIVED POLYNOMIALS .... 267—268
PROPERTIES OF DERIVED POLYNOMIALS .... 268—269
OF EQUAL ROOTS .... 269—272
IV. GENERAL ELIMINATION .... 272—277

CHAPTER XI.

SOLUTION OF NUMERICAL EQUATIONS.

I. PROPERTIES OF NUMERICAL EQUATIONS .... 277—279
II. LIMITS OF THE REAL ROOTS .... 279—284
III. COMMENSURABLE ROOTS .... 284—286
IV. STURM'S THEOREM .... 286—294
V. CUBIC EQUATIONS—CARDAN'S RULE .... 294—296
VI. HORNER'S METHOD .... 296—297

INTRODUCTION.

QUANTITY is a general term applicable to everything which can be increased or diminished, and measured.

MATHEMATICS is the science of quantity; that is, it is the science which treats of the measurement of quantities, and of their relations to each other. It is divided into two parts:

1st. The Pure Mathematics, embracing the principles of the science and all explanations of the processes by which these principles are derived from the quantities, Number and Space: and,

2d. The Mixed Mathematics, embracing the applications of the Pure Mathematics in the Mechanic Arts, and in all investigations involving the laws of matter, force, and motion.*

There are three operations of the mind which are immediately concerned in the investigations of mathematical science: 1st. Apprehension; 2d. Judgment; 3d. Reasoning.

1st. APPREHENSION is the notion, or conception of an idea in the mind, analogous to the perception by the senses.

---

* Davies' Nature and Utility of Mathematics. Book II.

2d. JUDGMENT is the comparing together, in the mind, two of the ideas which are the objects of Apprehension, and pronouncing that they agree or disagree with each other. Judgment, therefore, is either affirmative or negative.

3d. REASONING is the act of proceeding from one judgment to another, or of deducing unknown truths from principles already known. Language affords the *signs* by which these operations of the mind are expressed and communicated. An *apprehension*, expressed in language, is called a *term;* a *judgment*, expressed in language, is called a *proposition;* and a *process of reasoning*, expressed in language, is called a *demonstration.*[*]

The reasoning processes, in Logic, are conducted, usually, by means of words, and in all complicated cases, can take place in no other way. The words employed are *signs of ideas.* They are also among the principal instruments or helps of thought; and any imperfection in the instruments, or in the mode of using them, will destroy all ground of confidence in the result. So, in the Science of Mathematics, the meaning of the terms employed are accurately defined, while the language arising from the use of the symbols, in each branch, has a definite and precise signification.

---

[*] Davies' Nature and Utility of Mathematics. Book I.

In Arithmetic, the ten characters, called figures, are the alphabet of the language; the combinations of these characters constitute the pure language of the science; and the principles of numbers which are unfolded by means of this language, constitute the science itself.

In Geometry, the signs which are employed to indicate the boundaries and forms of portions of space, are simply the straight line and the curve; and these, in connection with our common language, make up the language of Geometry. This science treats of space, by comparing portions of it with each other, for the purpose of pointing out their properties and mutual relations.

Analysis is a general term embracing that entire portion of mathematical science in which the quantities considered are represented by letters of the alphabet, and the operations to be performed on them are indicated by signs.

Algebra, which is a branch of Analysis, is also a species of universal arithmetic, in which letters and signs are employed to abridge and generalize all processes involving numbers. It is divided into two parts, corresponding to the science and art of Arithmetic:

1st. That which has for its object the investigation of the properties of numbers, embracing all the processes of reasoning by which new properties are inferred from known ones; and,

2d. The solution of all problems or questions involving the determination of certain numbers which are unknown, from their connection with certain others which are known or given.

In Arithmetic, all quantity is regarded as made up of parts, which can be numbered exactly or approximatively. Propositions, therefore, concerning numbers, have this remarkable peculiarity, that they are propositions concerning all quantities which can be numbered. Algebra extends the generalization still further.

A number is a collection of things of the same kind, without reference to the nature or kind of thing, and is generally expressed by figures. Algebraic symbols may stand for *all numbers*, or for all quantities which numbers represent, or even for quantities which cannot be exactly expressed numerically.

In Geometry, each geometrical figure stands for a class; and when we have demonstrated a property of a figure, that property is considered proved for every figure of the class. In Algebra, all numbers, all lines, all surfaces, all volumes, may be denoted by a single symbol, a or x. Hence, the conclusions deduced by means of those symbols are true of all things whatever, and not like those of Number and Geometry, true only for particular classes of things. The symbols of Algebra, therefore, should not excite in our minds ideas of particular things. The written characters, a, b, c, d, x, y, z, serve as the representatives of things in general, whether abstract or concrete, whether known or unknown, whether real or imaginary, finite or infinite.

In the various uses which we make of these symbols, and the processes of reasoning carried on by means of them, the mind insensibly comes to regard them as *things*, and not as mere signs; and we constantly predicate of them the properties of things in general, without pausing to inquire what kind of thing is implied. All this we are at liberty to do, since the symbols being the representatives of quantity in general, there is no necessity of keeping the idea of *quantity* continually alive in the mind; and the processes of thought may, without danger, be allowed to rest on the symbols themselves, and therefore, become to that extent, merely mechanical. But when we look back and see on what the reasoning is based, and how the processes have been conducted, we shall find that every step was taken on the supposition that we were actually dealing with *things*, and not with *symbols;* and that without this understanding of the language, the whole system is without signification, and fails.*

The quantities which are the subjects of the algebraic analysis may be divided into two classes: those which are known or given, and those which are unknown or sought. The known are uniformly represented by the first letters of the alphabet, a, b, c, d, &c.; and the unknown by the final letters, x, y, z, v, &c.

Six operations, only, can be performed upon a quantity that will give results differing from the quantity itself, viz.:

---

* **Davies' Nature and Utility of Mathematics. Chap. IV.**

1st. To add a quantity to it;
2d. To subtract a quantity from it;
3d. To multiply it;
4th. To divide it;
5th. To raise it to any power;
6th. To extract a root of it.

Six signs, only, are employed to denote these operations. They are too well known to be repeated here. These, with the signs of equality and inequality, together with the letters of the alphabet, are the elements of the Algebraic Language.

The interpretation of the language of Algebra is the first thing to which the attention of a pupil should be directed; and he should be drilled in the meaning and import of the symbols, until their significations and uses are as familiar as the sounds of the letters of the alphabet.

All the apprehensions, or elementary ideas, of the Science of Algebra, are conveyed to the mind by means of definitions and arbitrary signs; and every judgment is the result of a comparison of such impressions. Hence, the connection between the symbols and the ideas which they stand for, should be so close and intimate, that the one shall always suggest the other; and thus, the processes of Algebra become chains of thought, in which each link fulfils the double office of a distinct and connecting proposition.

ELEMENTS OF ALGEBRA.

CHAPTER I.

DEFINITIONS AND INTRODUCTORY REMARKS.

1. **Quantity** is anything that can be increased, diminished, and measured; as number, distance, and weight.

A thing can be *measured* when it can be expressed in terms of another thing of the same kind, taken as a standard of comparison. This standard is called a *unit*, or *unit of measure.*

The *value* of a quantity is an expression for that quantity in terms of some assumed unit.

2. **Mathematics** is the science which treats of the properties, measurement, and relations of quantities.

3. **Algebra** is that branch of mathematics in which the quantities considered are denoted by letters, and the operations to be performed upon them are indicated by signs. The letters and signs are called *symbols.*

4. **Kinds of Quantities.**—The quantities treated of in Algebra are of two kinds:

1°. *Known quantities,* those whose values are given; and

2°. *Unknown quantities,* those whose values are required.

Known quantities are represented by figures only, or by leading letters of the alphabet; as, a, b, c, etc.

Unknown quantities are denoted by final letters; as, x, y, z, etc. Letters used to denote quantities are often written with one or more dashes; as a', b'', x''', etc.; these symbols are read, *a prime, b second, x third,* etc.

The symbols 1°, 2°, 3°, etc., are read *first, second, third,* etc.

5. Signs.—The signs employed in Algebra are of three kinds: 1°. Signs of *operation;* 2°. Signs of *relation;* and 3°. Signs of *abbreviation.*

6. Signs of Operation.—The *signs of operation* are the following:

1°. The *sign of addition,* +, called *plus.* When placed between two quantities, it indicates that the second is to be added to the first. Thus, the expression, $a + b$, read, *a plus b,* indicates that b is to be added to a.

2°. The *sign of subtraction,* —, called *minus.* When placed between two quantities, it indicates that the second is to be subtracted from the first. Thus, the expression, $a - b$, read, *a minus b,* indicates that b is to be subtracted from a.

3°. The *sign of multiplication,* ×. When placed between two quantities, it indicates that the first is to be multiplied by the second. Thus, the expression, $a \times b$, indicates that a is to be multiplied by b, or that a and b are to be multiplied together. The quantities to be multiplied together are called *factors,* and the result of the multiplication is called a *product.*

If more than two quantities are to be multiplied together, the result is called a *continued product.* When factors are represented by letters, they are called *literal factors.* In this case, the sign of multiplication may be replaced by a simple dot, or it may be omitted altogether. Thus, the continued product of a, b, and c may be represented by either of the expressions,

$$a \times b \times c, \qquad a \, . \, b \, . \, c, \qquad \text{or} \qquad a\,b\,c.$$

A *numerical* factor is one denoted by a number; thus, in the expression, $3ab$, 3 is a numerical factor.

4°. The *sign of division,* ÷. When written between two quantities, it indicates that the first is to be divided by the second. Thus, the expression, $a \div b$, indicates that a is to be divided by b. The same operation may be indicated by writing a over b, with a line between them, after the manner of a fraction in arithmetic; or, the sign of division may be replaced by a vertical bar, or by an arc. Thus, the expressions,

$$a \div b, \quad \frac{a}{b}, \quad a \underline{|b}, \quad \text{and} \quad b)a$$

are equivalent.

5°. The *exponential sign.* This sign consists of a number written to the right and above a quantity, to show how many times that quantity is taken as a factor. Thus, in the expressions, a^3 and a^n, 3 and n are exponents. In the former case, a is taken three times as a factor; and in the latter case, n times; that is, as many times as there are units in n. The resulting products are called *powers;* the expressions a^3 and a^n are read *a to the third power,* and *a to the* *n^{th} power.* When no exponent is *written,* the exponent 1 is always understood.

6°. The *radical sign,* $\sqrt{}$. When placed over a quantity, it indicates that some root of that quantity is to be extracted; that is, it indicates that the quantity is to be resolved into two or more equal factors. The nature of the root is indicated by a number written to the left and above the sign, called the *index.*

Thus, the expressions, $\sqrt[3]{a}$, $\sqrt[4]{a}$, or $\sqrt[n]{a}$, indicate that the *cube* root, the *fourth* root, or the n^{th} root of a is to be extracted; that is, that the quantity a is to be resolved into *three, four,* or n equal factors. When no index is *written,* the index 2 is always understood. Thus, the expression, $\sqrt{a}$, indicates that the *square root* of a is to be taken, or that a is to be resolved into two equal factors.

7. The signs + and —, besides indicating addition and subtraction, are also used to show the *sense* in which a quantity is taken. A quantity preceded by the sign +, is called *positive*, or additive; a quantity preceded by the sign —, is called *negative*, or subtractive.

The terms *positive* and *negative* are diametrically opposed to each other. If we agree to call a quantity *positive*, when taken in any sense, we must call it *negative* when taken in an opposite sense. Thus, if we agree to call distance estimated to the right, *positive*, distance estimated to the left must be *negative*. Hence, the sign —, when placed before a quantity, changes the *sense* in which the quantity is to be taken; that is, *it changes a positive quantity to a negative one, and a negative quantity to a positive one.*

The double sign, ±, read *plus* or *minus*, is used to show that the quantity is first to be taken *positively*, and then *negatively*. Thus, the expression, $a \pm b$, is equivalent to the two expressions, $a + b$ and $a - b$.

When no sign is written before a quantity, the sign + is always understood.

8. Signs of Relation.—The *signs of relation* are the following:

1°. The *sign of equality*, =. When written between two quantities, it indicates that they are equal to each other; that is, that they contain a given unit the same number of times. Thus, the expression,

$a = bc$, indicates that a is equal to the product of b and c.

2°. The *signs of inequality*, < and >. When either of these is written between two quantities, it indicates that they are unequal, the greater one being at the opening of the sign. Thus, the expressions,

$$a < b, \qquad \text{and} \qquad b > c,$$

indicate that a is *less* than b, and that b is *greater* than c.

3°. The *signs of proportion*, : :: :.

The single colon stands for, *is to;* the double colon for, *as*. Thus, the expression,

$$a \; : \; b \; :: \; c \; : \; d,$$

is read, *a is to b, as c is to d.* The double colon is equivalent to the sign of equality, and is often replaced by that sign.

4°. The *sign of variation*, $\propto$. When placed between two quantities, it indicates that they increase and decrease together. Thus, the expression, $x \propto yz$, indicates that x varies as the product of y and z; that is, that x, and yz increase and decrease together.

9. Signs of Abbreviation.—The *signs of abbreviation* are the following:

1°. The sign $\therefore$ stands for, *hence.*

2°. The *vinculum*, ———, the *bar*, |, the *parenthesis*, or *brackets*, ( ), [ ], { }, are employed to connect quantities which are to be operated on in the same manner. Thus, each of the expressions,

$$\overline{a+b+c}\times x, \qquad \left.\begin{array}{r} a \\ +b \\ +c \end{array}\right| x, \qquad (a+b+c)x,$$

$$[a+b+c]\,x \qquad \text{and} \qquad \{a+b+c\}\,x$$

indicates that the sum of a, b, and c is to be multiplied by x.

Other signs of abbreviation will be explained in their proper places.

10. An Algebraic Expression is the expression of a quantity in *algebraic language;* that is, by means of the symbols just explained.

Thus, $3ab$ is the algebraic expression of *the quantity formed by taking three times the product of the quantities denoted by a and b;* or more simply, *three times the product of a and b;* $2ab - c^2$ is the algebraic expression of *twice the product of a and b, diminished by the square of c.* The former expression is *simple;* the latter, *complex.*

11. A Coefficient is a number written before a quantity to show how many times it is to be taken additively. Thus, the expression, $a^2 + a^2 + a^2$, may be written $3a^2$, in which 3 is the coefficient of a^2; it is sometimes convenient to speak of 3 as the coefficient of the expression $3a^2$.

A coefficient may be *numerical* or *literal, simple* or *complex*. Thus, in the expressions, $3x^2$, $3abx^2$, and $(a + b)\ x^2$, the quantities 3, $3ab$, and $(a + b)$, are coefficients of x^2. If a quantity be resolved into two factors, either factor may be regarded as the coefficient of the other. Thus, in the expression, a^2b^3, we may regard a^2 as the coefficient of b^3; or we may regard b^3 as the coefficient of a^2; or we may regard a^2b as the coefficient of b^2; and so on. When no coefficient is written, the coefficient 1 is always understood.

12. Terms.—A quantity that can be written without the aid of either of the signs, + or —, is called a *term*. Thus, $3a^2$, $\frac{a}{b}$, and $c\sqrt{a}$, are terms.

A term may be preceded by either of the signs + or —; in this case, the sign is called *the sign of the term*, and it is used to show the *sense* in which the term is taken. Thus, the expression, $+ a^2$, shows that a^2 is taken positively; and the expression, $- a^2$, shows that a^2 is taken negatively.

Terms are classed according to the number of *literal factors* they contain. Terms that contain but *one* literal factor are of the *first degree;* terms that contain *two* literal factors are of the *second degree,* and so on. The numerical factor is not counted in estimating the degree of a term. Thus, a, $3b$, and $5d$, are terms of the first degree; ac, $4ab$, and $5ad$, are terms of the second degree.

In the expression, $4a^2b^3c$, a is taken *twice* as a factor, b is taken *three* times, and c is taken *once;* hence, the term is of the 6th degree. In like manner, it may be shown that the *degree* of any term is indicated by the *sum* of the exponents of all its letters.

Terms are *homogeneous* when they are of the same degree. Thus, $3ab$ and $4c^2$ are homogeneous; a^2xy and $3b^3c$ are also homogeneous.

Terms are similar when they have a common unit; that is, when they have the same literal part. Thus, the terms, $3a^2b$ and $-7a^2b$, are similar; the terms $3a \times x^2y$ and $4c \times x^2y$ are similar if we regard x^2y as the unit of both. In this case, $3a$ and $4c$ are to be regarded as coefficients of x^2y.

13. A **Monomial** is a single term unconnected with any other by the signs $+$ or $-$. Thus, $3a^2b$ and $-7ab^2$ are monomials.

Every monomial is made up of three elements:

1°. A *literal part*, which may be regarded as the *unit*;

2°. A *coefficient*, which shows how many times that unit is taken; and

3°. A *sign*, which shows in what sense it is taken. Thus, in the monomial $-7a^2b^3d$, we may regard the literal part, a^2b^3d, as the unit; the coefficient 7 shows that this unit is taken 7 times; and the sign, $-$, shows that $7a^2b^3d$ is taken in a negative sense.

14. A **Polynomial** is an expression composed of two or more terms, connected by the signs $+$ or $-$. Thus, the expression, $2a^2 - 3bc + 4b^2$, is a polynomial.

A *binomial* is a polynomial of *two* terms. Thus, $a^2 - 2b$ is a binomial.

A *trinomial* is a polynomial of three terms. Thus, $a^2 - 3a + y^2$ is a trinomial.

15. **The Reciprocal** of a quantity is 1 divided by that quantity. Thus, $\frac{1}{a}$ is the reciprocal of a. If the product of two quantities is equal to 1, each quantity is the reciprocal of the other. Thus, if $ab = 1$, then is $a = \frac{1}{b}$ and $b = \frac{1}{a}$; that is, each is the reciprocal of the other.

We know from arithmetic that $\frac{3}{7} \times \frac{7}{3} = 1$; hence $\frac{3}{7}$ is the reciprocal of $\frac{7}{3}$, and the reverse; that is, the reciprocal of the fraction $\frac{3}{7}$ is found by inverting its terms. In like manner, it may be shown that the reciprocal of any fraction is found by inverting its terms.

16. The Numerical Value of an algebraic expression is the result obtained by giving a numerical value to each letter that enters it, and then performing all the indicated operations. Thus, the numerical value of the expression,

$4ab - cd$, when $a = 2$, $b = 8$, $c = 2$, and $d = 12$,

is, $4 \times 2 \times 8 - 2 \times 12 = 64 - 24 = 40.$

Find the numerical values of the following expressions, when $a = 4$, $b = 2$, $c = 3$, and $d = 1$:

1. $12c + 3a - 5b - 2d.$ *Ans.* 36.
2. $\frac{5a}{b+c} - \frac{b}{c-d} + \frac{a}{b+2d}.$ *Ans.* 4.
3. $4ab - \frac{3cd}{b+d} - \frac{8a}{5c+1}.$ *Ans.* 27.
4. $\frac{6d^3}{c^2} - \frac{16b^2}{a} + \frac{12c^2}{5d^4} + a.$ *Ans.* $10\frac{4}{15}$.

Also, of the following when $w = 2$, $x = 3$, $y = 4$, and $z = 5$:

5. $(w^2 + x^2 + y^2 + z^2)(x^2 + y^2).$ *Ans.* 1350.
6. $w^2 + z^2 + y^2 + z^2(x^2 + y^2).$ *Ans.* 670.
7. $w^2 + (x^2 + y^2 + z^2)(x^2 + y^2).$ *Ans.* 1254.

Also, of the following when $x = 1$, $y = 2$, and $z = 3$:

8. $\frac{x^2 + y^2 + z^2}{z - x} \times \frac{z^2 - x^2}{2y}.$ *Ans.* 14.
9. $(x+y)^2(x^2-xy+y^2) + (x+z)^2(x^2-xz+z^2)$ *Ans.* 139.
10. $x^2(y^2 + z^2) + y^2(x^2 + z^2) + z^2(x^2 + y^2).$ *Ans.* 98.

The numerical value is independent of the *order* of the terms.

CHAPTER II.

FUNDAMENTAL OPERATIONS.

I. ADDITION.

17. Addition is the operation of finding the simplest equivalent expression for two or more algebraic quantities. This expression is called their *sum.*

In Arithmetic, the *sum* is greater than any of its parts, since all the numbers added have the same sign. In Algebra, the signs of the quantities to be added may be different—some *positive* and some *negative;* and hence, it may happen that their *aggregate,* or *sum,* may be less than any one of the parts. This aggregate is called the *algebraic sum,* to distinguish it from the *arithmetical sum.*

18. Similar Quantities.—When the quantities to be added are similar—that is, when they have a common unit—the operations indicated may be actually performed. Thus, the sum of $7a^2b$ and $4a^2b$ is $11a^2b$, because 7 units and 4 units are 11 units; and similarly in the following examples:

$$
\begin{array}{cccc}
(1.) & (2.) & (3.) & (4.) \\
3a^2x & +\ 9acb & -\ 2axy & -\ 5a^3y^2z^4 \\
5a^2x & +\ 6acb & -\ 3axy & -\ 9a^3y^2z^4 \\
6a^2x & +\ 3acb & -\ 4axy & -\ 7a^3y^2z^4 \\
\hline
14a^2x & +18acb & -\ 9axy & -21a^3y^2z^4
\end{array}
$$

19. To deduce a rule for the addition of similar terms, let us take the following examples:

$$\begin{array}{rcr}
(1.) & \qquad\qquad & (2.) \\
+\,3a^2xy^3 & & +\,4a^2xy^3 \\
-\,2a^2xy^3 & & -\,5a^2xy^3 \\
+\,6a^2xy^3 & & +\,3a^2xy^3 \\
-\,3a^2xy^3 & & -\,8a^2xy^3 \\
\hline
+\,4a^2xy^3 & & -\,6a^2xy^3
\end{array}$$

In these examples the common unit is a^2xy^3. In the *first*, this unit is taken *positively* $3 + 6$, or 9 times; and *negatively*, $2 + 3$, or 5 times; hence, the aggregate, or sum, is equal to $+ 9a^2xy^3 - 5a^2xy^3$, which may be written under the form

$$+ 4a^2xy^3 + 5a^2xy^3 - 5a^2xy^3;$$

but the last two terms are numerically equal and taken in an *opposite* sense; they therefore destroy each other—that is, their sum is equal to 0, and, therefore, there remains for the required sum, the quantity $+ 4a^2xy^3$.

In the *second*, the unit is taken 7 times positively and 13 times negatively; hence, the sum is $+ 7a^2xy^3 - 13a^2xy^3$, which can be written,

$$+ 7a^2xy^3 - 7a^2xy^3 - 6a^2xy^3;$$

but the first two terms of this result destroy each other, and we have for the required sum the quantity $- 6a^2xy^3$.

In each case, the coefficient of the required sum is numerically equal to the difference between the sums of the coefficients of the positive and negative terms, taken separately; and in both, it has the sign of the greater; and because all other groups of similar terms may be added, in like manner, we have the following

RULE.

Add the coefficients of the positive and negative terms separately; subtract the less sum from the greater, prefixing the sign of the greater; to the result annex the common unit.

This rule enables us to reduce a polynomial containing similar terms to its simplest form.

EXAMPLES.

1. Reduce the polynomial $4a^2b - 8a^2b - 9a^2b + 11a^2b$ to its simplest form. *Ans.* $-2a^2b$.

2. Reduce the polynomial $7abc^2 - abc^2 - 7abc^2 - 8abc^2 + 6abc^2$ to its simplest form. *Ans.* $-3abc^2$.

3. Reduce the polynomial $9cb^3 - 8ac^2 + 15cb^3 + 8ca + 9ac^2 - 24cb^3$ to its simplest form. *Ans.* $ac^2 + 8ca$.

4. Reduce the polynomial $6ac^2 - 5ab^3 + 7ac^2 - 3ab^3 - 13ac^2 + 18ab^3$ to its simplest form. *Ans.* $10ab^3$.

5. Reduce the polynomial $abc^2 - abc + 5ac^2 - 9abc^2 + 6abc - 8ac^2$ to its simplest form. *Ans.* $-8abc^2 + 5abc - 3ac^2$.

6. Reduce the polynomial $3a^2b^2 - 7a^3b + 5ab - 9a^2b^2 + 9a^3b + 3ab$ to its simplest form. *Ans.* $-6a^2b^2 + 2a^3b + 8ab$.

7. Reduce the polynomial $3acb^4 - 7a^3c^2b^3 - 6a^4b^5 - 3acb^4 + 6a^3c^2b^3 - 6acb^4 + 4a^4b^5 + 2a^4b^5$ to its simplest form.
Ans. $-a^3c^2b^3 - 6acb^4$.

8. Reduce the polynomial $-7a^2b^2c^3 + 9a^5bc^2 + 6a^2b^2c^3 + a^2b^2c^3 - 5a^5bc^2 - b^5c^5$ to its simplest form. *Ans.* $4a^5bc^2 - b^5c^5$.

9. Reduce the polynomial $-10a^3b + 6a^2b^2 + 7a^3b - 5a^2b^2 - 5a^3b + 3a^2b^2$ to its simplest form. *Ans.* $-8a^3b + 4a^2b^2$.

20. Dissimilar Quantities.—When the quantities to be added are dissimilar—that is, when they have no common unit—the operation can only be indicated. Thus, the sum of $3a$, $5b$, and $2c$, may be expressed by the polynomial,

$$3a + 5b - 2c;$$

and this result cannot be reduced to a simpler form. Hence, for the addition of polynomials, we have the following

RULE.

I. Write the quantities to be added so that similar terms shall fall in the same column:

II. Add each column separately, annexing dissimilar terms, with their proper signs.

EXAMPLES.

1. Add the polynomials, $3a^2 - 2b^2 - 4ab$, $5a^2 - b^2 + 2ab$, and $3ab - 3c^2 - 2b^2$.

By the preceding rule, we have

(1.)

$$\begin{array}{lrrrr}
 & 3a^2 & -\ 4ab & -\ 2b^2 & \\
 & 5a^2 & +\ 2ab & -\ b^2 & \\
 & & +\ 3ab & -\ 2b^2 & -\ 3c^2 \\
\hline
\text{Sum}\ \ldots\ldots & 8a^2 & +\ ab & -\ 5b^2 & -\ 3c^2
\end{array}$$

Each term, when disposed of, should be cancelled, that none may be omitted.

(2.)

$$\begin{array}{lrrr}
 & 7x & +\ 3ab & +\ 2c \\
 & -\ 3x & -\ 3ab & -\ 5c \\
 & 5x & -\ 9ab & -\ 9c \\
\hline
\text{Sum} & 9x & -\ 9ab & -\ 12c
\end{array}$$

(3.)

$$\begin{array}{rrr}
16a^2b^2 & +\ bc & -\ 2abc \\
-\ 4a^2b^2 & -\ 9bc & +\ 6abc \\
9a^2b^2 & +\ bc & +\ abc \\
\hline
3a^2b^2 & -\ 7bc & +\ 5abc
\end{array}$$

(4.)

$$\begin{array}{lrrrr}
 & a & +\ ab & -\ cd & +\ f \\
 & 3a & +\ 5ab & -\ 6cd & -\ f \\
 & -\ 5a & -\ 6ab & +\ 6cd & -\ 7f \\
 & -\ a & +\ ab & +\ cd & +\ 4f \\
\hline
\text{Sum} & -\ 2a & +\ ab & +\ 0 & -\ 3f
\end{array}$$

(5.)

$$\begin{array}{rrrr}
6ab & +\ cd & +\ d & \\
3ab & +\ 5cd & -\ y & \\
-\ 4ab & +\ 6cd & +\ x & \\
-\ 5ab & -\ 12cd & +\ y & \\
\hline
 & & +\ x & +\ d
\end{array}$$

6. Add the polynomials $3a+b$, $3a+3b$, $-9a-7b$, $6a+9b$, and $8a + 3b + 8c$. *Ans.* $11a + 9b + 8c$.

7. Add the polynomials $3ax + 3ac + f$, $-9ax + 7a + d$, $+6ax + 3ac + 3f$, $8ax + 13ac + 9f$, and $-14f + 3ax$.

Ans. $11ax + 19xc - f + 7a + d$.

8. Add the polynomials $3a^2c + 5ab$, $7a^2c - 3ab + 3ac$, $5a^2c - 6ab + 9ac$, and $-8a^2c + ab - 12ac$. *Ans.* $7a^2c - 3ab$.

9. Add the polynomials $19a^2x^3b - 12a^3cb$, $5a^2x^3b + 15a^3cb - 10ax$, $-2a^2x^3b - 13a^3cb$ and $-18a^2x^3b - 12a^3cb + 9ax$.
Ans. $4a^2x^3b - 22a^3cb - ax$.

10. Add together, $3a + b + c$, $5a + 2b + 3ac$, $a + c + ac$, and $-3a - 9ac - 8b$. *Ans.* $6a - 5b + 2c - 5ac$.

11. Add the polynomials $5a^2b + 6cx + 9bc^2$, $7cx - 8a^2b$, and $-15cx - 9bc^2 + 2a^2b$. *Ans.* $-a^2b - 2cx$.

12. Add together, $8ax + 5ab + 3a^2b^2c^2$, $-18ax + 6a^2 + 10ab$, and $10ax - 15ab - 6a^2b^2c^2$.
Ans. $-3a^2b^2c^2 + 6a^2$.

13. What is the sum of $41a^3b^2c - 27abc - 14a^2y$ and $10a^3b^2c + 9abc$? *Ans.* $51a^3b^2c - 18abc - 14a^2y$.

14. What is the sum of $18abc - 9ab + 6c^2 - 3c + 9cx$, and $9abc + 3c - 9ax$? *Ans.* $27abc - 9ab + 6c^2$.

15. What is the sum of $8abc + b^3a - 2cx - 6xy$ and $7cx - xy - 13b^3a$? *Ans.* $8abc - 12b^3a + 5cx - 7xy$.

16. What is the sum of $9a^2c - 14aby + 15a^2b^2$ and $-a^2c - 8a^2b^2$? *Ans.* $8a^2c - 14aby + 7a^2b^2$.

17. What is the sum of $17a^5b^2 + 9a^3b - 3a^2$, $-14a^5b^2 + 7a^2 - 9a^3$, $-15a^3b + 7a^5b^2 - a^3$, and $14a^3b - 19a^3b$?
Ans. $10a^5b^2 - 11a^3b + 4a^2 - 10a^3$.

18. What is the sum of $3ax^2 - 7ax^3 - 17axy$, $+9ax^2 + 18ax^3 + 34axy$, and $7a^5b + 3ax^3 - 7ax^2 + 4bcx$?
Ans. $5ax^3 + 9ax^3 + 17axy - 7ax^2 + 4bcx$.

19. Add, $3a^2 + 5a^2b^2c^2 - 9a^3x$, $7a^2 - 8a^2b^2c^2 - 10a^3x$, and $10ab + 16a^2b^2c^2 + 19a^3x$. *Ans.* $10a^2 + 13a^2b^2c^2 + 10ab$.

20. Add, $7a^2b - 3abc - 8b^2c - 9c^3 + cd^2$, $8abc - 5a^2b + 3c^3 - 4b^2c + cd^2$, and $4a^2b - 8c^3 + 9b^2c - 3d^3$.
Ans. $6a^2b + 5abc - 3b^2c - 14c^3 + 2cd^2 - 3d^3$.

21. Add, $-18a^3b + 2ab^4 + 6a^2b^2$, $-8ab^4 + 7a^3b - 5a^2b^2$, and $-5a^3b + 6ab^4 + 11a^2b^2$. *Ans.* $-16a^3b + 12a^2b^2$.

22. What is the sum of $3a^3b^2c - 16a^4x - 9ax^3d$, $+6a^3b^2c - 6ax^3d + 17a^4x$, and $+16ax^3d - a^4x - 8a^3b^2c$? *Ans.* $a^3b^2c + ax^3d$.

23. What is the sum of the following terms: viz., $8a^5 - 10a^4b - 16a^3b^2 + 4a^2b^3 - 12a^4b + 15a^3b^2 + 24a^2b^3 - 6ab^4 - 16a^3b^2 + 20a^2b^3 + 32ab^4 - 8b^5$?
Ans. $8a^5 - 22a^4b - 17a^3b^2 + 48a^2b^3 + 26ab^4 - 8b^5$.

24. Add the polynomials $12y - 7ax - 5a\sqrt{x}$, $5ax + a\sqrt{x} - 8y$, $-ax - y + 9a\sqrt{x}$, $5y + 4ax - 3a\sqrt{x}$, and $-ax + a\sqrt{x} + y$. *Ans.* $9y + 3a\sqrt{x}$.

25. Find the sum of $a^3 - \frac{3}{4}a^2x + \frac{1}{3}ax^2 - \frac{5}{8}x^3$, $a^3 + \frac{1}{2}a^2x + \frac{1}{4}ax^2 + \frac{2}{3}x^3$, and $2a^3 - \frac{1}{4}a^2x + \frac{7}{12}ax^2 + \frac{1}{24}x^3$.
Ans. $4a^3 - \frac{1}{2}a^2x + \frac{14}{12}ax^2 + \frac{1}{12}x^3$.

II. SUBTRACTION.

21. **Subtraction** is the operation of finding from *two* quantities a *third*, which being added to the *second* will produce the *first*.

The *first* is called the *minuend;* the *second*, the *subtrahend;* and the *third*, the *difference*, or *remainder*.

In arithmetic the remainder is always less than the minuend; in algebra it may happen that the remainder is greater than the minuend. Hence, in algebra, the difference between two quantities is called the *algebraic difference*, to distinguish it from the *arithmetical difference*.

22. To deduce a rule for subtraction, let us take from $4ab$, $2x - 3ay + b$. The operation is indicated thus,

$$4ab - (2x - 3ay + b);$$

the sign —, before the parenthesis, shows that the sign of the subtrahend is to be changed (Art. **7**); but changing the sign of the subtrahend is the same as changing the sign of each term from + to —, or from — to +; making this change, we have,

$$4ab - 2x + 3ay - b.$$

In this case, we were to subtract from $4ab$, the *difference* between $2x + b$ and $3ay$. Changing the signs of $2x + b$, to minus, we take away *too much* by $3ay$, which should, therefore, be added; and since all other cases may be like treated, we have the following

RULE.

Change the sign of every term of the subtrahend, or conceive it to be changed, and then proceed as in addition.

When the minuend and subtrahend contain similar terms, it will be convenient to write them down so that similar terms shall fall in the same column.

EXAMPLES.

1. From $6ac - 5ab + c^2$ subtract $3ac + 3ab - 7c$.

OPERATION.

$$\begin{array}{lllll}
\text{Minuend . . .} & 6ac & - 5ab & + c^2 & \\
\text{Subtrahend . .} & 3ac & + 3ab & & - 7c \\
\hline
\text{Remainder . .} & 3ac & - 8ab & + c^2 & + 7c
\end{array}$$

Changing the sign of $3ac$ from + to — and adding it to $6ac$, we have $3ac$; in like manner, changing the sign of $3ab$ and adding it to $-5ab$, we have $-8ab$; changing the sign of $-7c$ and adding it to the dissimilar term $+c^2$, we have $c^2 + 7c$.

This result may be verified by adding the *remainder* to the subtrahend; the sum will be found equal to the minuend. Every example in subtraction may be verified in like manner

	(2.)	(3.)
From . .	$16a^2 - 5bc + 7ac$	$19abc - 16ax - 5axy$
Subtract .	$14a^2 + 5bc + 8ac$	$17abc + 7ax - 15axy$
Remainder	$2a^2 - 10bc - ac$	$2abc - 23ax + 10axy$

	(4.)	(5.)
From . .	$5a^3 - 4a^2b + 3b^2c$	$4ab - cd + 3a^2$
Subtract .	$-2a^3 + 3a^2b - 8b^2c$	$5ab - 4cd + 3a^2 + 5b^2$
Remainder	$7a^3 - 7a^2b + 11b^2c.$	$-ab + 3cd + 0 - 5b^2$

6. From $3a^2x - 13abc + 7a^2$, subtract $9a^2x - 13abc$.
Ans. $-6a^2x + 7a^2$.

7. From $51a^3b^2c - 18abc - 14a^2y$, subtract $41a^3b^2c - 27abc - 14a^2y$.
Ans. $10a^3b^2c + 9abc$.

8. From $27abc - 9ab + 6c^2$, subtract $9abc + 3c - 9ax$.
Ans. $18abc - 9ab + 6c^2 - 3c + 9ax$.

9. From $8abc - 12b^3a + 5cx - 7xy$, subtract $7cx - xy - 13b^3a$.
Ans. $8abc + b^3a - 2cx - 6xy$.

10. From $8a^2c - 14aby + 7a^2b^2$, subtract $9a^2c - 14aby + 15a^2b^2$.
Ans. $-a^2c - 8a^2b^2$.

11. From $9a^6x^2 - 13 + 20ab^3x - 4b^6cx^2$, subtract $3b^6cx^2 + 9a^6x^2 - 6 + 3ab^3x$.
Ans. $17ab^3x - 7b^6cx^2 - 7$.

12. From $5a^4 - 7a^3b^2 - 3c^5d^2 + 7d$, subtract $3a^4 - 3a^2 - 7c^5d^2 - 15a^3b^2$.
Ans. $2a^4 + 8a^3b^2 + 4c^5d^2 + 7d + 3a^2$.

13. From $51a^2b^2 - 48a^3b + 10a^4$, subtract $10a^4 - 8a^3b - 6a^2b^2$.
Ans. $57a^2b^2 - 40a^3b$.

14. From $21x^3y^2 + 25x^2y^3 + 68xy^4 - 40y^5$, subtract $64x^2y^3 + 48xy^4 - 40y^5$.
Ans. $20xy^4 - 39x^2y^3 + 21x^3y^2$.

15. From $53x^3y^2 - 15x^2y^3 - 18x^4y - 56x^5$, subtract $-15x^2y^3$ $18x^3y^2 + 24x^4y$.
Ans. $35x^3y^2 - 42x^4y - 56x^5$.

16. From $10a^m - 15b^n - c^p + 5d^q$, subtract $-9a^m + 2b^n + c^p - 5d^q$. *Ans.* $19a^m - 17b^n - 2c^p + 10d^q$.

17. From $\frac{3}{2}a + 36 + \frac{1}{4}c + \frac{1}{2}bc$, subtract $\frac{5}{2}a - 76 - 3bc + \frac{1}{2}c$. *Ans.* $-a + 112 - \frac{1}{4}c + \frac{7}{2}bc$.

18. From $\frac{1}{2}y - \frac{5}{2}a - \frac{3}{4}x + \frac{1}{3}a$, subtract $3y + \frac{11}{4}a - \frac{2}{3}x$. *Ans.* $-\frac{5}{2}y - \frac{59}{12}a - \frac{1}{12}x$.

23. Brackets.—The sign + before a parenthesis or brackets, shows that the signs of all the included terms are to remain unchanged. Hence, in this case, the parenthesis or brackets may be dropped. Thus,

$$a + (b + c - d) = a + b + c - d.$$

The sign — before a parenthesis or brackets, shows that the signs of all the included terms are to be *changed* (Art. **22**). Hence, in this case, we may drop the parenthesis or brackets, provided we change the signs of all the included terms. Thus,

$$6a^2 - (3ab - 2p^2) = 6a^2 - 3ab + 2p^2$$

Conversely, any number of terms of a polynomial may be placed within a parenthesis, preceded by the sign —, provided we change the sign of each term so included. Thus,

$$2b^3 + 7a^3 - 8a^2b - 4bc = 2b^3 - (-7a^3 + 8a^2b + 4bc).$$

EXAMPLES.

Let the following expressions be transformed so as to get rid of the parentheses and brackets, and then let them be reduced to their simplest form:

1. $3x - 7y - (x + 2y) - (4y - 7x)$. *Ans.* $9x - 13y$.

2. $(a+b+c)-(a+b-c)-(a-b-c)$. *Ans.* $-a+b+3c$.

3. $x^2 - 2xy + y^2 - (x^2 + xy - y^2) - (-2xy + x^2 + y^2)$. *Ans.* $-x^2 - xy + y^2$.

4. $a^2 - (b^2 - c^2) - b^2 + (c^2 - a^2) + c^2 - (b^2 - a^2)$. *Ans.* $a^2 - 3b^2 + 3c^2$.

In cases involving double brackets, the subordinate ones are to be got rid of first. Thus,

5. $1-[1-(2-x)]+[4x-(3-6x)]+[4-(-5+6x)]$
$=1-[1-2+x]+[4x-3+6x]+[4+5-6x]$
$=1-1+2-x+4x-3+6x+4+5-6x=8+3x.$

6. $2a-[2a-(b+2x)]+[b-(2x-2b)]$. *Ans.* $4b$.

7. $[(1+x)-(1+2x)]+[(1-x)+(1-2x)]-[(1-x)-(1-2x)]$. *Ans.* $2-5x$.

8. $a^2+2ab+b^2-[a^2+ab-b^2-(2ab-a^2-b^2)]$.
Ans. $-a^2+3ab+b^2$.

9. $11x-\{7x-[8x-(9x-6x)]\}$. *Ans.* $9x$.

10. $4x-3y-\{(2x+4y)+3x+[y-9x-(2y-x)+(x-y)]\}$.
Ans. $6x-5y$.

24. Essential Sign.—From the preceding principles, we have,

$$x+(+y)=x+y;$$
$$x+(-y)=x-y;$$
$$x-(+y)=x-y;$$
$$x-(-y)=x+y.$$

In these expressions, the sign of y within the parenthesis is called *the sign of the quantity;* the sign that precedes the parenthesis is called *the sign of operation;* and the resulting, or final sign of y, is called *the essential sign.* By inspecting the given examples, it will be seen that when the sign of the quantity and the sign of operation are *alike*, the essential sign is $+$; when the sign of the quantity and the sign of operation are *unlike*, the essential sign is $-$.

If in each expression we write a before the parenthesis, it is plain that the principle will not be changed, and we shall have,

$$x+a(+y)=x+ay;$$
$$x+a(-y)=x-ay;$$
$$x-a(+y)=x-ay;$$
$$x-a(-y)=x+ay.$$

III. MULTIPLICATION.

25. Multiplication is the operation of finding the *product* of two quantities.

The quantity to be multiplied is called the *multiplicand;* the quantity by which it is multiplied is called the *multiplier;* and each is called a *factor* of the product.

26. From Article **24** we infer:

1°. If a positive quantity be taken any number of times positively, or if a negative quantity be taken any number of times negatively, the result will be positive; that is, *the product of two terms having like signs is plus.*

2°. If a positive quantity be taken any number of times negatively, or if a negative quantity be taken any number of times positively, the result will be negative; that is, *the product of two terms having unlike signs is minus.* Hence, the sign of the product of any two terms may be determined by the following

RULE.

Like signs in the two factors give + in the product, and unlike signs give −.

27. Monomials.—To deduce a rule for the multiplication of monomials, let us find the product of $7a^3b^2$ and $4a^2bc$. The operation may be indicated thus:

$$7a^3b^2 \times 4a^2bc;$$

or, resolving both multiplicand and multiplier into their simple factors,

$$7aaabb \times 4aabc.$$

It has been shown in arithmetic that the value of the product is not changed by changing the order of its factors; hence, we may write the product as follows:

$$7 \times 4aaaaabbbc = 28a^5b^3c.$$

Comparing this with the given monomials, we see that the coefficient of the product is the *product* of the coefficients of the multiplicand and multiplier, and that the exponent of each letter in the product is equal to the *sum* of the exponents of that letter in both factors; and since the same course of reasoning may be applied to any two monomials, we have the following

RULE.

I. Multiply the coefficients together for a new coefficient:

II. After this, write all the letters in both factors, giving to each an exponent equal to the sum of its exponents in the two factors.

The sign of the product is to be determined by the rule for signs.

EXAMPLES.

(1.) . . . $8a^2bc^2 \times 7abd^2 = 56a^3b^2c^2d^2.$

(2.) . . . $21a^3b^2dc \times 8abc^3 = 168a^4b^3c^4d.$

	(3.)	(4.)	(5.)	(6.)
Multiply . .	$3a^2b$	$12a^2x$	$6xyz$	a^2xy
by . .	$2ba^2$	$12x^2y$	ay^2z	$2xy^2$
	$6a^4b^2$	$144a^2x^3y$	$6axy^3z^2$	$2a^2x^2y^3.$

7. Multiply $8a^5b^2c$ by $7a^8b^5cd.$ *Ans.* $56a^{13}b^7c^2d.$

8. Multiply $5abd^3$ by $12cd^5.$ *Ans.* $60abcd^8.$

9. Multiply $7a^4bd^2c^3$ by $abdc.$ *Ans.* $7a^5b^2d^3c^4.$

10. Multiply $6x^2y^2z^2$ by $-\frac{1}{2}ay^3z.$ *Ans.* $-3ax^2y^5z^3.$

11. Multiply $-19a^2x^2$ by $-\frac{1}{6}a^3x.$ *Ans.* $\frac{19}{6}a^5x^3.$

12. Multiply $26x^{n}y^{m}$ by $-2x^{m}y^{n}.$ *Ans.* $52x^{m+n}y^{m+n}.$

13. Multiply $-5a^3y^2$ by $3ay^{n}z.$ *Ans.* $15a^4y^{n+2}z.$

14. Multiply $-2x^my^2$ by $-3x^3y^pz$. *Ans.* $6x^{m+3}y^{p+2}z$.

15. Multiply $4a^mx^ny^p$ by $-4a^nx^3y^2$.
Ans. $-16a^{m+n}x^{n+3}y^{p+2}$.

28. Continued Product.—The preceding rule may be extended to finding the continued product of any number of monomials. The sign of the product will then depend on the number of negative monomials employed. If this number is *even*, the sign of the product will be +; if the number is *uneven*, the sign will be —. Hence, every even power of a negative quantity is +, and every odd power is —.

EXAMPLES.

Find the continued product of the following groups of monomials:

16. $2a^2xy$, $3ayx^2$, and $-4abxy$. *Ans.* $-24a^4bx^4y^3$.

17. $3bx^2$, $-2y^2z$, and $-3yz^3$. *Ans.* $18bx^2y^3z^4$.

18. max^p, $-3ay^q$, and $-na^2y^p$. *Ans.* $3mna^4x^py^{p+q}$.

19. $\frac{1}{2}x^2y^m$, $-\frac{1}{4}x^py^q$, $-3x^qyz$, and $-2a^2x$.
Ans. $-\frac{3}{4}a^2x^{p+q+3}y^{m+q+1}z$.

20. $20a^px$, $\frac{1}{5}ax^p$, and $-\frac{1}{4}x^qy$. *Ans.* $-a^{p+1}x^{p+q+1}y$.

21. $-\frac{1}{4}x^2y$, $-\frac{1}{4}x^2y$, and $-\frac{1}{4}x^2y$.
Ans. $(-\frac{1}{4}x^2y)^3 = -\frac{1}{64}x^6y^3$.

22. $-\frac{1}{2}xy^2$, $-\frac{1}{2}xy^2$, $-\frac{1}{2}xy^2$, and $-\frac{1}{2}xy^2$.
Ans. $(-\frac{1}{2}xy^2)^4 = \frac{1}{16}x^4y^8$.

The degree of the product of two or more monomials is denoted by the sum of the numbers that denote the degrees of the factors taken separately. Thus, in example 16th, each factor is of the 4th degree, and the product of the three factors is of the 12th degree.

29. A Polynomial by a Monomial.—If all the terms of any polynomial be multiplied by the same quantity, the aggregate of the products will be equal to the product of the polynomial by that quantity. Hence, we have the following rule for multiplying a polynomial by a monomial:

RULE.

Multiply every term of the polynomial by the monomial, and connect the results by their proper signs.

If the multiplicand is in its simplest form, the product will also be in its simplest form.

EXAMPLES.

(1.)

Multiplicand . . . $3ax - 2ay + z$
Multiplier . . . . $4a$
Product. . . . . $12a^2x - 8a^2y + 4az$

(2.)

Multiplicand . . . $5x^3 - 2x^2 + x - 4$
Multiplier . . . $-15ax$
Product . . . . $-75ax^4 + 30ax^3 - 15ax^2 + 60ax$

Find the products of the following:

3. $2x^3yz - 3x^2y^2z + 5xyz^3$, and $-7x^3yz$.
Ans. $-14x^6y^2z^2 + 21x^5y^3z^2 - 35x^4y^2z^4$.

4. $13x^ny^m - 2x^my^n$, and $-3x^2y^3$.
Ans. $-39x^{n+2}y^{m+3} + 6x^{m+2}y^{n+3}$.

5. $a^mc^n + b^ma^n - c^mb^n$, and a^{2m}.
Ans. $a^{3m}c^n + b^ma^{2m+n} - a^{2m}c^mb^n$.

6. $\frac{5}{2}x^2y - \frac{3}{4}xy^3 + y$, and $-\frac{1}{4}x^2y^3$.
Ans. $-\frac{5}{8}x^4y^4 + \frac{3}{16}x^3y^6 - \frac{1}{4}x^2y^4$.

If the monomial be taken as the multiplicand, and the ynomial as the multiplier, the product will remain un- ed.

30. A Polynomial by a Polynomial.—To multiply one polynomial by another, we have, from what precedes, the following

RULE.

Multiply all the terms of the multiplicand by each term of the multiplier, and reduce the polynomial result to its simplest form.

EXAMPLES.

(1.)

$$\begin{array}{lllll}
\text{Multiply . .} & 3a^2 & + 4ab & + b^2 & \\
\text{by . . .} & 2a & + 5b & & \\
\hline
 & 6a^3 & + 8a^2b & + 2ab^2 & \\
 & & + 15a^2b & + 20ab^2 & + 5b^3 \\
\hline
\text{Product . .} & 6a^3 & + 23a^2b & + 22ab^2 & + 5b^3
\end{array}$$

(2.)

$$\begin{array}{llll}
x^2 + y^2 & & & \\
x - y & & & \\
\hline
x^3 + xy^2 & & & \\
 & - x^2y & - y^3 & \\
\hline
x^3 + xy^2 & - x^2y & - y^3 &
\end{array}$$

(3.)

$$\begin{array}{llll}
x^5 & + xy^6 & + 7ax & \\
ax & + 5ax & & \\
\hline
ax^6 & + ax^2y^6 & + 7a^2x^2 & \\
 & + 5ax^6 & + 5ax^2y^6 & + 35a^2x^2 \\
\hline
6ax^6 & + 6ax^2y^6 & + 42a^2x^2 &
\end{array}$$

4. Multiply $x^2 + 2ax + a^2$ by $x + a$.
Ans. $x^3 + 3ax^2 + 3a^2x + a^3$.

5. Multiply $x^2 + y^2$ by $x + y$.
Ans. $x^3 + xy^2 + x^2y + y^3$.

6. Multiply $3ab^2 + 6a^2c^2$ by $3ab^2 + 3a^2c^2$.
Ans. $9a^2b^4 + 27a^3b^2c^2 + 18a^4c^4$.

7. Multiply $4x^2 - 2y$ by $2y$. *Ans.* $8x^2y - 4y^2$.

8. Multiply $2x + 4y$ by $2x - 4y$. *Ans.* $4x^2 - 16y^2$.

9. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$.
Ans. $x^4 - y^4$.

10. Multiply $x^2 + xy + y^2$ by $x^2 - xy + y^2$.
Ans. $x^4 + x^2y^2 + y^4$.

In order to bring similar terms of the product together, it is found convenient to arrange both polynomials with respect to the same letter; that is, we write them down so that the first term shall contain the highest power of that letter, the second term the next highest power, and so on to the last. The letter, with respect to which the arrangement is made, is called the *leading* letter.

11. Multiply $4a^3 - 5a^2b - 8ab^2 + 2b^3$
by $2a^2 - 3ab - 4b^2$

$$\begin{array}{rrrrrr}
8a^5 & - 10a^4b & - 16a^3b^2 & + 4a^2b^3 & & \\
 & - 12a^4b & + 15a^3b^2 & + 24a^2b^3 & - 6ab^4 & \\
 & & - 16a^3b^2 & + 20a^2b^3 & + 32ab^4 & - 8b^5 \\
\hline
8a^5 & - 22a^4b & - 17a^3b^2 & + 48a^2b^3 & + 26ab^4 & - 8b^5
\end{array}$$

In this example the polynomials are arranged with respect to a; a simple inspection of the operation shows the advantage of the arrangement. It will be observed that the *product* is also arranged with respect to the same letter

12. Multiply $2a^2 - 3ax + 4x^3$, by $5a^2 - 6ax - 2x^2$.
Ans. $10a^4 - 27a^3x + 34a^2x^2 - 18ax^3 - 8x^4$.

13. Multiply $3x^2 - 2yx + 5$, by $x^2 + 2xy - 3$.
Ans. $3x^4 + 4x^3y - 4x^2 - 4x^2y^2 + 16xy - 15$.

14. Multiply $3x^3 + 2x^2y^2 + 3y^2$, by $2x^3 - 3x^2y^2 + 5y^3$.
Ans. $\left\{\begin{array}{l} 6x^6 - 5x^5y^2 - 6x^4y^4 + 6x^3y^2 + 15x^3y^3 \\ - 9x^2y^4 + 10x^2y^5 + 15y^5. \end{array}\right.$

15. Multiply $8ax - 6ab - c$, by $2ax + ab + c$.
Ans. $16a^2x^2 - 4a^2bx - 6a^2b^2 + 6acx - 7abc - c^2$.

16. Multiply $3a^2 - 5b^2 + 3c^2$, by $a^2 - b^3$.
Ans. $3a^4 - 5a^2b^2 + 3a^2c^2 - 3a^2b^3 + 5b^5 - 3b^3c^2$.

17. Multiply $3a^2 - 5bd + cf$,
by $-5a^2 + 4bd - 8cf$.

Product, $-15a^4 + 37a^2bd - 29a^2cf - 20b^2d^2 + 44bcdf - 8c^2f^2$

18. Multiply $4a^3b^2 - 5a^2b^2c + 8a^2bc^2 - 3a^2c^3 - 7abc^3$,

by $2ab^2 - 4abc - 2bc^2 + c^3$.

$$\text{Product}\;..\begin{cases} \;8a^4b^4 - 10a^3b^4c + 28a^3b^3c^2 - 34a^3b^2c^3 \\ -\;4a^2b^3c^3 - 16a^4b^3c + 12a^3bc^4 + 7a^2b^2c^4 \\ +\;14a^2bc^5 + 14ab^2c^5 - 3a^2c^6 - 7abc^6. \end{cases}$$

19. Multiply $x^2 - \frac{x}{3} + \frac{1}{4}$, by $\frac{x}{2} + \frac{1}{4}$

Ans. $\frac{1}{2}x^3 + \frac{1}{12}x^2 + \frac{1}{24}x + \frac{1}{16}$.

20. Multiply $x + \frac{1}{2a^2}x^3 - \frac{1}{4a^2}x^3$, by $1 - \frac{1}{2a}x + \frac{1}{4a^2}x^2$.

Ans. $x - \frac{1}{4a^2}x^3 + \frac{1}{4a^3}x^4 - \frac{1}{16a^4}x^5$.

21. Multiply $a^m + 3b^n - 2c^p$, by $a^m - 3b^n + 2c^p$.

Ans. $a^{2m} - 9b^{2n} + 12b^nc^p - 4c^{2p}$.

22. Find the continued product of $x - 1$, $x + 1$, $x^2 - 1$, and $x^2 + 1$. *Ans.* $x^6 - x^4 - x^2 + 1$.

23. Find the continued product of $x^2 + x + 1$, $x^2 - x + 1$, and $x^2 + x - 1$. *Ans.* $x^6 + x^5 + x^3 + x - 1$.

24. Find the continued product of $x + 3$, $x + 5$, and $x + 7$. *Ans.* $x^3 + 15x^2 + 71x + 105$.

31. Results.—The following conclusions are deducible from the principles of multiplication:

1°. *If both multiplicand and multiplier are homogeneous, the product will be homogeneous, and the degree of any term of the product will be indicated by the sum of the numbers that indicate the degrees of its two factors.*

Thus, in example 18th, each term of the multiplicand is of the 5th degree, each term of the multiplier of the 3d degree, and each term of the product of the 8th degree.

2°. *If no two terms of the product are similar, there will be no reduction amongst them; and the number of terms in* *the product will then be equal to the number of terms in the multiplicand, multiplied by the number of terms in the multiplier.*

This is evident, since *each* term of the multiplier will produce as many terms as there are terms in the multiplicand. Thus, in example 16, there are *three* terms in the multiplicand, *two* in the multiplier, and *six* in the product.

3°. *Among the terms of the product there are always two that cannot be reduced with any others.*

With reference to any letter common to both factors, the irreducible terms are:

1°. *The term produced by the multiplication of the two terms of the multiplicand and multiplier that contain the highest power of this letter; and*
2°. *The term produced by the multiplication of the two terms that contain the lowest power of this letter.*

For, these partial products will contain this letter, to a higher and to a lower power than either of the other partial products, and consequently, cannot be similar to any of them.

32. A formula is an algebraic expression of a general rule or principle.

33. Useful Formulas.—The following formulas depending on the rule for multiplication, will be found useful in practical operations:

Let a and b represent any two quantities; then $a + b$ will represent their sum, and $a - b$ their difference.

From the definition of a square, we have,

$$(a + b)^2 = (a + b) \times (a + b).$$

Performing the multiplication indicated, we have,

$$(a + b)^2 = a^2 + 2ab + b^2 \quad . \; . \; . \; . \; . \; . \quad (1)$$

Hence,

1°. *The square of the sum of any two quantities is equal to the square of the first,* plus *twice the product of the first by the second,* plus *the square of the second.*

Thus, $(5a^2 + 8a^2b)^2 = (5a^2)^2 + 2 \times 5a^2 \times 8a^2b + (8a^2b)^2$;

or, $(5a^2 + 8a^2b)^2 = 25a^4 + 80a^4b + 64a^4b^2$.

We have also,

$$(a - b)^2 = (a - b) \times (a - b).$$

Performing the operation indicated, we have,

$$(a - b)^2 = a^2 - 2ab + b^2 \quad . \; . \; . \; . \; . \quad (2)$$

Hence,

2°. *The square of the difference of any two quantities* is equal to *the square of the first,* minus *twice the product of the first by the second,* plus *the square of the second.*

Thus, $(7ab - 12c)^2 = (7ab)^2 - 2 \times 7ab \times 12c + (12c)^2$;

or, $(7ab - 12c)^2 = 49a^2b^2 - 168abc + 144c^2$.

We also have, by actual multiplication,

$$(a + b)(a - b) = a^2 - b^2 \quad . \; . \; . \; . \quad (3)$$

Hence,

3°. *The product of the sum of any two quantities by their difference is equal to the difference of their squares.*

Thus, $(3a + 4c)(3a - 4c) = 9a^2 - 16c^2$.

Formulas (1), (2), and (3) are much used in factoring.

It is to be observed that these formulas are perfectly general in their nature—that is, they are entirely independent of the values of the quantities considered. They are algebraic expressions of the law of formation of the products considered.

Verify the following formulas by actual multiplication, and observe their law of formation:

$$(x+a)(x+b)=x^2+(a+b)x+ab \quad . \; . \; . \quad (4)$$

$$(x^2+xy+y^2)(x-y)=x^3-y^3 \quad . \; . \; . \; . \; . \quad (5)$$

$$(x^2-xy+y^2)(x+y)=x^3+y^3 \quad . \; . \; . \; . \; . \quad (6)$$

IV. DIVISION.

34. **Division** is the operation of finding from two quantities a *third*, which being multiplied into the *second* will produce the *first*.

The first quantity is called the *dividend*, the second is called the *divisor*, and the third is called the *quotient*. The divisor and quotient are *factors* of the dividend, and the dividend is said to be a *multiple* of both.

35. **Monomials.**—In dividing one monomial by another, such a sign must be given to the quotient as will make the sign of the product of the quotient and divisor the same as that of the dividend. Hence, when the dividend and divisor have *like signs*, the quotient must be +; when the dividend and divisor have *unlike* signs, the quotient must be —. Hence, we may write the following rule for signs in division:

RULE.

Like signs give + in the quotient, and unlike signs give —.

36. **Division of Monomials.**—To deduce a rule for the division of monomials. Find the quotient of $35a^5b^2c^2$ by $7a^2bc$.

The operation may be indicated thus,

$$\frac{35a^5b^2c^2}{7a^2bc} = \text{quotient, } 5a^3bc.$$

For, the quotient must be such a quantity as multiplied by the divisor will produce the dividend: hence, the coefficient of the quotient multiplied by 7 must give 35; therefore, it is 5. The exponent of each letter in the quotient must be such that, when added to the exponent of the same letter in the divisor, the sum will be equal to the exponent of that letter in the dividend. Hence, the exponent of a in the quotient is 3, the exponent of b is 1, that of c is 1, and the required quotient is $5a^3bc$.

Since we may reason in a similar manner upon any two monomials, we have for the division of monomials the following

RULE.

I. Divide the coefficient of the dividend by that of the divisor, for a new coefficient:

II. Write after this coefficient all the letters in the dividend, giving to each an exponent equal to the excess of its exponent in the dividend over that in the divisor.

The sign of the quotient is determined by the rule for signs.

EXAMPLES.

1. Divide $16x^2$ by $8x$. *Ans.* $2x$.
2. Divide $15a^2xy^3$ by $3ay$. *Ans.* $5axy^2$.
3. Divide $84ab^3x$ by $12b^2$. *Ans.* $7abx$.
4. Divide $-96a^4b^2c^3$ by $12a^2bc$. *Ans.* $-8a^2bc^2$.
5. Divide $144a^9b^8c^7d^5$ by $-36a^4b^6c^6d$. *Ans.* $-4a^5b^2cd^4$.
6. Divide $-256a^3bc^2x^3$ by $-16a^2cx^2$. *Ans.* $16abcx$.
7. Divide $-300a^5b^4c^3x^2$ by $30a^4b^3c^2x$. *Ans.* $-10abcx$.
8. Divide $-400a^8b^6c^4x^5$ by $25a^8b^5c^3x$. *Ans.* $-16bcx^4$.

9. Divide $-7a^{m}x^{n}$ by $-a^{n}x^{m}$. *Ans.* $+7a^{m-n}x^{n-m}$.

10. Divide $25a^{3}b^{2}c^{m}$ by $-5a^{3}bc^{n}$. *Ans.* $-5bc^{m-n}$.

37. Zero Power.—When the exponent of any letter is the same in both dividend and divisor, that letter may be dropped from the quotient, or it may be retained with 0 for an exponent. For, by the rule for division, we have,

$$\frac{a^{m}}{a^{m}} = a^{m-m} = a^{0}; \quad \text{but } \frac{a^{m}}{a^{m}} = 1; \quad \therefore \quad a^{0} = 1.$$

From this we infer, *that the* 0 *power of any quantity is equal to* 1, *and the reverse.* Hence, any quantity may be introduced into a monomial, as a *factor*, provided we give it the exponent 0.

38. Division Exact.—In the division of monomials, the quotient is said to be *entire* when its coefficient is *a whole number*, and when the exponents of all the letters that enter it are *positive;* in this case the division is *exact.* Hence, in order that the division may be exact,

1°. The coefficient of the dividend must be exactly divisible by that of the divisor; and

2°. The exponent of each letter that enters the dividend must be equal to, or greater than, the exponent of that letter in the divisor.

In all other cases the quotient is essentially fractional, and the division is *inexact.* In such cases, the indicated quotient may often be simplified by canceling all the factors that are common to both dividend and divisor. Thus,

$$\frac{12a^{8}b^{6}c^{7}}{16a^{7}b^{5}c^{9}} = \frac{3ab \times 4a^{7}b^{5}c^{7}}{4c^{2} \times 4a^{7}b^{5}c^{7}} = \frac{3ab}{4c^{2}}.$$

By applying the rule, we have,

$$\frac{12a^{8}b^{6}c^{7}}{16a^{7}5^{5}c^{9}} = \frac{3}{4}abc^{-2},$$

in which the exponent of c is negative.

39. Negative Exponents.—To explain the meaning of negative exponents, we have, by the rule,

$$\frac{a^2}{a^5} = a^{-3}; \quad \text{but} \quad \frac{a^2}{a^5} = \frac{1}{a^3}; \quad \therefore \quad a^{-3} = \frac{1}{a^3}.$$

Hence, we infer that, *a quantity with a negative exponent is the reciprocal of the same quantity with an equal positive exponent.* We also infer that a factor may be transferred from the *numerator* of a fraction to the *denominator*, or the *reverse*, by *changing the sign* of its exponent.

According to the definition, a *positive* exponent shows how many times a quantity is taken as a factor (Art. **6**, 5°.). From what has just been shown, a *negative* exponent shows how many times it is taken as a divisor. This result corresponds to the principle explained in Article **7**, viz: that a quantity, having a negative sign, is to be understood in a sense directly opposite to what is implied when it is preceded by a positive sign.

ADDITIONAL EXAMPLES.

11. Divide $12a^{m}x^{n}y^{2}$ by $7a^{2m}x^{n}y^{3}$. *Ans.* $\frac{12}{7}a^{-m}x^{0}y^{-1}$.

12. Divide $-11a^{3}b^{2}x^{4}$ by $2a^{2}b^{3}x$. *Ans.* $-\frac{11}{2}ab^{-1}x^{3}$.

13. Divide $3x^{-5}y^{2}z$ by $-5xy^{2}z^{2}$. *Ans.* $-\frac{3}{5}x^{-6}z^{-1}$.

14. Divide $mx^{-a}y^{b}$ by $nx^{-c}y^{-1}$. *Ans.* $\frac{m}{n}x^{c-a}y^{b+1}$.

15. Divide $-7mx^{-4}y^{+4}$ by $-3x^{-2}y^{-1}$. *Ans.* $\frac{7}{3}mx^{-2}y^{5}$.

16. Divide $4x^{m+n}y^{m-n}$ by $2x^{m}y^{n}$. *Ans.* $2x^{n}y^{m-2n}$.

40. Polynomial by Monomial.—The quotient of a polynomial by a monomial is equal to the algebraic sum of the quotients obtained by dividing every term of the polynomial by the monomial. Hence, to divide a polynomial, we have the following

RULE.

Divide each term of the dividend by the divisor, and connect the quotients by their proper signs.

EXAMPLES.

1. Divide $6a^2x^4y^6 - 12a^3x^3y^6 + 15a^4x^5y^3$ by $3a^2x^2y^3$.
Ans. $2x^2y^4 - 4axy^4 + 5a^2x^3y$.

2. Divide $12a^4y^6 - 16a^5y^5 + 20a^6y^4 - 28a^7y^3$ by $-4a^4y^3$.
Ans. $-3y^3 + 4ay^2 - 5a^2y + 7a^3$.

3. Divide $15a^2bc - 20acy^2 + 5cd^2$ by $-5abc$.
Ans. $-3a + \frac{4y^2}{b} - \frac{d^2}{ab}$.

4. Divide $6a^{4m} - 12a^{3m}x^m + 3x^mx^{2m}$ by $3a^{m-1}x^m$.
Ans. $2a^{3m+1}x^{-m} - 4a^{2m+1} + ax^m$.

5. Divide $2x^{2m} - 3x^{2(m-1)} + 6x^{2(m-2)}$ by $3x^{m-2}$.
Ans. $\frac{2}{3}x^{m+2} - x^m + 2x^{m-2}$.

6. Divide $-a^{2m} - \frac{3}{2}a^{m+n-p}$ by $-\frac{1}{2}a^m$.
Ans. $2a^m + 3a^{n-p}$.

41. Polynomials.—To deduce a rule for the division of polynomials, let us find the quotient of

$$26a^2b^2 + 10a^4 - 48a^3b + 24ab^3 \text{ by } 4ab - 5a^2 + 3b^2.$$

Let both polynomials be arranged with respect to the same letter, say a, and, for convenience of multiplication, as well as for economy of space, let the divisor be written at the right of the dividend, as shown below:

$$\begin{array}{lr|l}
 & \textit{Dividend.} & \textit{Divisor.} \\
 & 10a^4 - 48a^3b + 26a^2b^2 + 24ab^3 & \underline{-5a^2 + 4ab + 3b^2} \\
 & \underline{+10a^4 - 8a^3b - 6a^2b^2 } & -2a^2 + 8ab \\
\text{1st rem.} & -40a^3b + 32a^2b^2 + 24ab^3 & \textit{Quotient.} \\
 & \underline{-40a^3b + 32a^2b^2 + 24ab^3.} & \\
\text{2d rem.} & 0 &
\end{array}$$

From what was shown in Art. **31**, we know that the first term of the arranged quotient must be such that its product by the first term of the divisor shall produce the first term of the dividend; hence, if we divide the first term of the dividend by the first term of the divisor, we have $-2a^2$ for the first term of the quotient. The product of the divisor by this term of the quotient gives

$$+10a^4 - 8a^3b - 6a^2b^2,$$

which being subtracted from the dividend, gives a new dividend, on which we operate as before; and so on, till a remainder is found which is equal to 0, or whose first term is not exactly divisible by the first term of the divisor. Since all similar cases may be treated in the same way, we have the following

RULE.

I. *Arrange both dividend and divisor with reference to the same letter: then divide the first term of the dividend by the first term of the divisor, for the first term of the quotient; multiply the divisor by this term, and subtract the product from the dividend:*

II. *Divide the first term of the remainder by the first term of the divisor, for the second term of the quotient; multiply the divisor by this term, and subtract the product from the first remainder. Continue this operation till a remainder is found equal to* 0, *or whose first term is not exactly divisible by the first term of the divisor.*

In performing the division it is not necessary to bring down all the terms of the dividend to form the several remainders, but they may be brought down successively, as in the following

EXAMPLE.

Divide $x^4 - 4x^3 - 34x^2 + 76x + 105$ by $x - 7$.

$$
\begin{array}{l|l}
\qquad\qquad\textit{Dividend.} & \qquad\textit{Divisor.} \\
x^4 - 4x^3 - 34x^2 + 76x + 105 & x - 7 \\ \cline{2-2}
\underline{x^4 - 7x^3} & x^3 + 3x^2 - 13x - 15 \\
\qquad 3x^3 - 34x^2 & \qquad\textit{Quotient.} \\
\qquad \underline{3x^3 - 21x^2} & \\
\qquad\qquad -13x^2 + 76x & \\
\qquad\qquad \underline{-13x^2 + 91x} & \\
\qquad\qquad\qquad\quad -15x + 105 & \\
\qquad\qquad\qquad\quad \underline{-15x + 105} & \\
\qquad\qquad\qquad\qquad\quad 0 &
\end{array}
$$

When a remainder is found equal to 0, the division is *exact*. When a remainder is found whose first term is not divisible by the first term of the divisor, the exact division is impossible. In this case we write after the quotient, already found, the last remainder, placing the divisor under it, after the manner of a fraction in arithmetic.

EXAMPLE.

Divide $x^4 - 16x^2 + 3$ by $x^2 - 4$.

$$
\begin{array}{lr|ll}
 & \textit{Dividend.} & \textit{Divisor.} & \\
 & x^4 - 16x^2 + 3 & x^2 - 4 & \\ \cline{3-3}
 & \underline{x^4 - 4x^2\phantom{{}+3}} & x^2 - 12 & \textit{Partial Quotient.} \\
\text{1st remainder . .} & -12x^2 + 3 & & \\
 & \underline{-12x^2 + 48} & & \\
\text{2d remainder . . . . . .} & -45 & &
\end{array}
$$

Hence, the entire quotient is,

$$x^2 - 12 - \frac{45}{x^2 - 4},$$

and the division is not exact.

It may happen that the division can be carried on indefinitely. In this case we may stop at any stage of the operation, and terminate the quotient, as just shown.

EXAMPLE.

Divide 1 by $1 - x$.

$$
\begin{array}{lll}
 & \textit{Dividend.} & \textit{Divisor.} \\
 & 1 & \big|\,1 - x \\ \cline{3-3}
 & \underline{1 - x} & \;1 + x + x^2 \quad \textit{Partial Quotient.} \\
\text{1st remainder . .} & x & \\
 & \underline{x - x^2} & \\
\text{2d remainder . . .} & x^2 & \\
 & \underline{x^2 - x^3} & \\
\text{3d remainder . . . .} & x^3 &
\end{array}
$$

Stopping at the third remainder, we have for the entire quotient

$$1 + x + x^2 + \frac{x^3}{1 - x}.$$

GENERAL EXAMPLES.

1. Divide $10ab + 15ac$ by $5a$. *Ans.* $2b + 3c$.
2. Divide $30ax - 54x$ by $6x$. *Ans.* $5a - 9$.
3. Divide $10x^2y - 15y^2 - 5y$ by $5y$. *Ans.* $2x^2 - 3y - 1$.
4. Divide $12a + 3ax - 18ax^2$ by $3a$. *Ans.* $4 + x - 6x^2$.
5. Divide $6ax^2 + 9a^2x + a^2x^2$ by ax. *Ans.* $6x + 9a + ax$.
6. Divide $a^2 + 2ax + x^2$ by $a + x$. *Ans.* $a + x$.
7. Divide $a^3 - 3a^2y + 3ay^2 - y^3$ by $a - y$. *Ans.* $a^2 - 2ay + y^2$.
8. Divide $24a^2b - 12a^3cb^2 - 6ab$ by $-6ab$. *Ans.* $-4a + 2a^2cb + 1$.
9. Divide $6x^4 - 96$ by $3x - 6$. *Ans.* $2x^3 + 4x^2 + 8x + 16$.

10. Divide $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$ by $a^2 - 2ax + x^2$. *Ans.* $a^3 - 3a^2x + 3ax^2 - x^3$.

11. Divide $48x^3 - 76ax^2 - 64a^2x + 105a^3$ by $2x - 3a$. *Ans.* $24x^2 - 2ax - 35a^2$.

12. Divide $y^6 - 3y^4x^2 + 3y^2x^4 - x^6$ by $y^3 - 3y^2x + 3yx^2 - x^3$. *Ans.* $y^3 + 3y^2x + 3yx^2 + x^3$.

13. Divide $64a^4b^6 - 25a^2b^8$ by $8a^2b^3 + 5ab^4$. *Ans.* $8a^2b^3 - 5ab^4$.

14. Divide $6a^3 + 23a^2b + 22ab^2 + 5b^3$ by $3a^2 + 4ab + b^2$. *Ans.* $2a + 5b$.

15. Divide $6ax^6 + 6ax^2y^6 + 42a^2x^2$ by $ax + 5ax$. *Ans.* $x^5 + xy^6 + 7ax$.

16. Divide $-15a^4 + 37a^2bd - 29a^2cf - 20b^2d^2 + 44bcdf - 8c^2f^2$ by $3a^2 - 5bd + cf$. *Ans.* $-5a^2 + 4bd - 8cf$.

17. Divide $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$. *Ans.* $x^2 + xy + y^2$.

18. Divide $x^4 - y^4$ by $x - y$. *Ans.* $x^3 + x^2y + xy^2 + y^3$.

19. Divide $3a^4 - 8a^2b^2 + 3a^2c^2 + 5b^4 - 3b^2c^2$ by $a^2 - b^2$. *Ans.* $3a^2 - 5b^2 + 3c^2$.

20. Divide $6x^6 - 5x^5y^2 - 6x^4y^4 + 6x^3y^2 + 15x^3y^3 - 9x^2y^4 + 10x^2y^5 + 15y^5$ by $3x^3 + 2x^2y^2 + 3y^2$. *Ans.* $2x^3 - 3x^2y^2 + 5y^3$.

21. Divide $8 + 12x^{-1} + 2x^{-2} + 2x^{-4}$ by $x^{-2} - 2x^{-1} + 4$. *Ans.* $2x^{-2} + 4x^{-1} + 2$.

22. Divide $x^{2n} - y^{2n}$ by $x^n + y^n$ *Ans.* $x^n - y^n$.

23. Divide $x^{3n} - 3x^{2n}y^n + 3x^ny^{2n} - y^{3n}$ by $x^n - y^n$. *Ans.* $x^{2n} - 2x^ny^n + y^{2n}$.

24. Divide $x^4 + x^{-4} - x^2 - x^{-2}$ by $x - x^{-1}$. *Ans.* $x^3 - x^{-3}$.

25. Divide $x^{m+1} + x^m y + xy^m + y^{m+1}$ by $x^m + y^m$.

Ans. $x + y$.

26. Divide $2a^{3n} - 6a^{2n}b^n + 6a^n b^{2n} - 2b^{3n}$ by $a^n - b^n$.

Ans. $2a^{2n} - 4a^n b^n + 2b^{2n}$.

27. Divide 1 by $1 + x$.

Ans. $1 - x + x^2 - x^3 + x^4 -$, etc.

28. Divide 1 by $1 - 2x + x^2$.

Ans. $1 + 2x + 3x^2 + 4x^3 +$, etc.

29. Divide $x - a$ by $x - b$.

Ans. $1 - (a - b)x^{-1} - (a - b)bx^{-2} - (a - b)b^2x^{-3} -$, etc.

30. Divide $x^4 - \dfrac{19x^2}{6} + \dfrac{x}{3} + \dfrac{1}{6}$ by $x^2 - 2x + \dfrac{1}{2}$.

Ans. $x^2 + 2x + \dfrac{1}{3}$.

42. Principles.—The following principles, easily deducible from what precedes, often enable us to determine by inspection when the *exact* division of one quantity by another is impossible:

1°. *When that term of the dividend which contains the highest power of any letter is not exactly divisible by that term of the divisor which contains the highest power of the same letter.*

For example, the polynomial $12a^3 - 5a^2b + 7ab^2 - 11b^3$ is not exactly divisible by $4a^2 + 8ab + 3b^2$; for, the term $-11b^2$ is not exactly divisible by $3b^2$. Inasmuch as the two polynomials might be arranged with respect to the ascending as well as with respect to the descending powers of any letter, it follows also that the exact division is impossible when that term of the dividend which contains the *lowest* power of any letter is not exactly divisible by that term of the divisor which contains the *lowest* power of the *same* letter.

2°. *When the divisor contains a letter that does not enter the dividend.*

For, it is impossible to multiply the divisor by any entire quantity that will cause that letter to disappear.

3°. *A monomial is not exactly divisible by a polynomial.*

For, the product of a polynomial by a monomial contains as many terms as the given polynomial, and the product of one polynomial by another contains at least two terms that are irreducible with each other (Art. **31**).

4°. *When the dividend contains a letter that does not enter the divisor, the exact division is impossible, unless the coefficients of the different powers of that letter are separately divisible by the divisor.*

Thus, the polynomial

$$(a^3 - b^3)x^2 + (a^2 - b^2)x + a - b,$$

is divisible by $a - b$ because the coefficients of the different powers of x are separately divisible by it; but the given polynomial is not divisible by $a + b$, because the coefficients of the different powers of x are not separately divisible by it. The quantity $a - b$ is said to be *independent of* **x**, as is also the quantity, $a + b$.

V. FACTORING.

43. Factoring is the operation of resolving a quantity into factors; that is, it is the operation of finding two or more quantities whose product is equal to a given quantity.

44. No general and definite rule can be laid down for ctoring algebraic expressions. The operation is usually per- ned by *inspection;* that is, by comparing the quantity to ctored with some other quantity of the same form, whose factors are known, and thence inferring the form of the required factors.

The formulas of Art. **33**, numbered from (1) to (6), are much used in factoring, as well as certain other formulas, depending on principles which will now be deduced.

45. First Principle.—*The difference of any like powers of two quantities is always exactly divisible by the difference of the quantities.*

Let a and b denote any two quantities, and m any positive whole number; then will $a^m - b^m$ denote the difference of any like powers of the quantities, and $a - b$ will denote the difference of the quantities. If we commence the division, we have the following

OPERATION.

$$\begin{array}{lc|l} & a^m - b^m & \underline{a - b} \\ & \underline{a^m - a^{m-1}b} & a^{m-1} \\ \text{Rem.} & a^{m-1}b - b^m & \end{array}$$

The first remainder can be written under the form, $b(a^{m-1} - b^{m-1})$; hence, we have,

$$\frac{a^m - b^m}{a - b} = a^{m-1} + b\,\frac{a^{m-1} - b^{m-1}}{a - b}$$

This quotient will be entire when $\dfrac{a^{m-1} - b^{m-1}}{a - b}$ is entire; that is, *if the difference of the* $(m-1)^{\text{th}}$ *powers of two quantities is divisible by the difference of the quantities, then will the difference of the* m^{th} *powers of the quantities be divisible by the difference of the quantities.*

But we know that the difference of the *second* powers of any two quantities is divisible by the difference of the quantities; hence, from the principle just demonstrated, *the difference of the third powers of two quantities is divisible by the difference of the quantities.* But the difference of their third powers being divisible by the difference of the quantities, it follows, from the same principle, that *the difference of their fourth powers is divisible by the difference of the quantities.* The difference of the fourth powers being divisible by the difference of the quantities, it follows that *the difference of their fifth powers is divisible by the difference of the quantities.* By continuing this process of successive deduction, it may be shown that the division is possible when m is any positive whole number whatever. Hence, *the principle is proved.*

The first term of the quotient is a^{m-1}. If we continue the division, we shall find the second term to be $a^{m-2}b$; the third term, $a^{m-3}b^2$; the fourth term, $a^{m-4}b^3$, and so on; that is, the coefficient of each term is equal to 1: the exponent of a diminishes by 1 and the exponent of b increases by 1, in each term toward the right. Hence, we may write the quotient as follows:

$$\frac{a^m - b^m}{a - b} = a^{m-1} + a^{m-2}b + a^{m-3}b^2 +, \text{etc.} \quad + ab^{m-2} + b^{m-1} \;\ldots\; (1)$$

Second Principle.—*The sum of like odd powers of any two quantities is exactly divisible by the sum of the quantities.*

If we suppose m to be an odd number, and replace a by c and b by $-d$, in the formula of the preceding principle, remembering that odd powers of a negative quantity are $-$, and even powers $+$, (Art. **28**), we have,

$$\frac{c^m + d^m}{c + d} = c^{m-1} - c^{m-2}d +, \text{etc.} \quad - cd^{m-2} + d^{m-1} \qquad (2)$$

But, the quotient in this case is entire; hence, *the principle is proved.*

If we make $m = 3$, in formula (1), we have,

$$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2 \qquad (3)$$

If we make $m = 3$, in formula (2), we have,

$$\frac{c^3 + d^3}{c + d} = c^2 - cd + d^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we make $a = c^2$, $b = d^2$, and $m = 2$, in formula (1), we have,

$$\frac{c^4 - d^4}{c^2 - d^2} = c^2 + d^2, \quad \text{or} \quad \frac{c^4 - d^4}{(c - d)\ (c + d)} = c^2 + d^2 \quad . \quad . \quad (5)$$

Formulas (3), (4), and (5) may, from the definition of division, be written under the following forms:

$$a^3 - b^3 = (a - b)\ (a^2 + ab + b^2) \quad . \quad . \quad . \quad . \quad (6)$$

$$c^3 + d^3 = (c + d)\ (c^2 - cd + d^2) \quad . \quad . \quad . \quad . \quad (7)$$

$$c^4 - d^4 = (c - d)\ (c + d)\ (c^2 + d^2) \quad . \quad . \quad . \quad (8)$$

The last three formulas are much used in factoring, and other formulas can be deduced in like manner.

46. When all the terms of a polynomial have a common monomial factor, that monomial is a factor of the polynomial, and the other factor may be found by dividing out the monomial factor. Thus, ab^2 is obviously a factor of every term of the polynomial,

$$ab^2c + 5ab^3 + ab^2c^2;$$

hence, the polynomial, when factored, may be written,

$$ab^2(c + 5b + c^2).$$

EXAMPLES.

1. Factor, $25a^4 - 30a^3b + 15a^2b^2$.

Ans. $5a^2(5a^2 - 6ab + 3b^2)$.

2. Factor, $3a^2b + 9a^2c + 18a^2xy$.

Ans. $3a^2(b + 3c + 6xy)$.

3. Factor, $8a^2cx - 18acx^2 + 2ac^5y - 30a^6c^9x$.

Ans. $2ac(4ax - 9x^2 + c^4y - 15a^5c^8x)$.

4. Factor, $24a^2b^2cx - 30a^8b^5c^6y + 36a^7b^8cd + 6abc$.
Ans. $6abc(4abx - 5a^7b^4c^5y + 6a^6b^7d + 1)$.

5. Factor, $3a^3b^{-2} - 6a^2b^{-3} + 12ab^{-4}c$.
Ans. $3ab^{-2}(a^2 - 2ab^{-1} + 4b^{-2}c)$.

6. Factor, $8x^{-2}y^{-3} - 12x^{-3}y^{-2} - 16x^{-4}y^{-1}$.
Ans. $4x^{-2}y^{-1}(2y^{-2} - 3x^{-1}y^{-1} - 4x^{-2})$.

47. By the aid of formulas (1), (2), and (3), of Art. **33**, we are enabled to factor the following expressions:

7. Factor, $9x^2 + 24xy + 16y^2$.
Ans. $(3x + 4y)(3x + 4y)$.

8. Factor, $4x^2y^2 - 4x^3y^4 + x^4y^6$.
Ans. $(2xy - x^2y^3)(2xy - x^2y^3)$.

9. Factor, $9x^2y^4 - 4x^6y^2$.
Ans. $(3xy^2 - 2x^3y)(3xy^2 + 2x^3y)$.

10. Factor, $a^2x^2 - b^2x^{-2}$.
Ans. $(ax + bx^{-1})(ax - bx^{-1})$.

11. Factor, $a^2y^{-2n} + 2aby^{-n}x^{-m} + b^2x^{-2m}$.
Ans. $(ay^{-n} + bx^{-m})(ay^{-n} + bx^{-m})$.

12. Factor, $18x^{-4} + 12x^{-3}y + 2x^{-2}y^2$.
Ans. $2x^{-2}(3x^{-1} + y)(3x^{-1} + y)$.

48. By the aid of formulas (4) and (5), of Art. **33**, we may factor the following expressions:

13. Factor, $x^2 + 17x + 70$. *Ans.* $(x + 7)(x + 10)$.

14. Factor, $x^2 - 17x + 70$. *Ans.* $(x - 7)(x - 10)$.

15. Factor, $x^2 + 3x - 70$. *Ans.* $(x - 7)(x + 10)$.

16. Factor, $x^2 - 3x - 70$. *Ans.* $(x + 7)(x - 10)$.

17. Factor, $x^3 - 8b^6$. *Ans.* $(x^2 + 2xb^2 + 4b^4)(x - 2b^2)$.

18. Factor, $x^6 - 27a^3$. *Ans.* $(x^4 + 3ax^2 + 9a^2)(x^2 - 3a)$.

19. Factor, $x^2 + 8x + 15$. *Ans.* $(x + 3)(x + 5)$.

20. Factor, $x^3 - 2x^2 - 15$. *Ans.* $x(x + 3)(x - 5)$.

49. By the aid of formulas (6), (7), and (8), of Art. **45**, we may factor the following expressions:

21. Factor, $27a^3 - 8b^3$.

Ans. $(3a - 2b)\ (9a^2 + 6ab + 4b^2)$.

22. Factor, $27x^3 + 64y^3$.

Ans. $(3x + 4y)\ (9x^2 - 12xy + 16y^2)$.

23. Factor, $a^4b^4 - c^4d^{-4}$.

Ans. $(ab - cd^{-1})\ (ab + cd^{-1})\ (a^2b^2 + c^2d^{-2})$.

24. Factor, $x^{-3} - y^3$.

Ans. $(x^{-1} - y)\ (x^{-2} + x^{-1}y + y^2)$.

25. Factor, $16a^4x^8 - y^8x^4$.

Ans. $(2ax^2 - y^2x)\ (2ax^2 + y^2x)\ (4a^2x^4 + y^4x^2)$.

26. Factor, $24b^6x^5 - 3x^2y^3$.

Ans. $3x^2(2b^2x - y)\ (4b^4x^2 + 2b^2xy + y^2)$.

50. In the following examples the mode of factoring is indicated:

27. $x^3-6x^2+11x-6=x^3-x^2-5x^2+5x+6x-6$
$=x^2(x-1)-5x(x-1)+6(x-1)=(x^2-5x+6)\ (x-1)$.

28. $x^8 + x^6y^2 + x^2y + y^3$
$= x^6(x^2 + y^2) + y(x^2 + y^2) = (x^6 + y)\ (x^2 + y^2)$.

29. $3x^3 - 3x^2y + xy^2 - y^3$
$= 3x^2(x - y) + y^2(x - y) = (3x^2 + y^2)\ (x - y)$.

51. In like manner the following expressions may be factored:

30. Factor, $10ax + 8ay + 15bx + 12by$.

Ans. $(5x + 4y)\ (2a + 3b)$.

31. Factor, $x^3 - 3x^2 + 3x - 2$.

Ans. $(x - 2)\ (x^2 - x + 1)$.

32. Factor, $x^3 - ax^2 - a^2x + a^3$. *Ans.* $(x^2 - a^2)\ (x - a)$.

VI. GREATEST COMMON DIVISOR.

52. A Common Divisor of two quantities is any quantity that will exactly divide each of them.

53. A Prime Factor of a quantity is a factor that cannot be resolved into other factors.

54. The Greatest Common Divisor of two quantities is the continued product of all the prime factors common to both.

It is called the *greatest* common divisor because it is that common factor which is of the *highest degree* with respect to the *exponents* of all the letters common to both quantities.

When both quantities can be factored, by inspection, the greatest common divisor may be found by the following

RULE.

I. Resolve the two quantities into prime factors:
II. Then find the continued product of all those factors that are common to both.

EXAMPLES.

1. Find the greatest common divisor of $ax(x^3 - a^3)$ and $a^2x - a^3$.

Factoring, $ax(x^3 - a^3) = ax(x - a)(x^2 + ax + a^2)$;

and $a^2x - a^3 = a^2(x - a) = aa(x - a)$.

Hence, by the rule, the greatest common divisor is

$$a(x - a).$$

Find the greatest common divisor of the following:

2. $x^2 - x - 2$, and $x^2 + 2x + 1$. *Ans.* $x + 1$.

3. $x^2 - a^2$, and $x^3 + a^3$. *Ans.* $x + a$.

4. $x^3 + 5x^2$, and $x^2 + 4x - 5$. *Ans.* $x + 5$.

5. $x^2 - 4x + 3$, and $x^2 - x - 6$. *Ans.* $x - 3$.

6. $x^2 + 2x + 1$, and $x^3 + 2x^2 + 2x + 1$. *Ans.* $x + 1$.

55. It is often impossible to factor the quantities by inspection. To deduce a rule for finding the greatest common divisor in all cases, it will be necessary to establish the following

Principle.—*Any quantity that is a common factor of two other quantities is also a factor of the difference between any two multiples of those quantities.*

Let P and P′ be any two quantities, both of which are divisible by a common factor, D; and let Q and Q′ be the corresponding quotients; then will $P = QD$ and $P' = Q'D$. Let m and n be any two multipliers whatever; then will $mP - nP'$ be the difference between any two multiples of P and P′. If we replace P and P′ by their values, QD and Q′D, and then factor, we have,

$$mP - nP' = mQD - nQ'D = (mQ - nQ')\,D.$$

This shows that D is a factor of $mP - nP'$, *which was to be proved.*

Hence, if a polynomial be multiplied by any multiplier, and if a second polynomial be taken from it any number of times, the remainder will contain all the factors common to the two polynomials; that is, if the first polynomial be multiplied by any multiplier, and the result be divided by the second polynomial, the remainder will contain all the factors common to the two polynomials.

In like manner, if the second polynomial, or any multiple of it, be divided by the first remainder, or any multiple of it, the second remainder will contain all the factors common to the polynomials used, and consequently to the given polynomials; and so on indefinitely.

Since m and n are any multipliers whatever, they may be either factors or divisors; that is, we may either introduce or suppress monomial factors at any stage of the operation, provided we do not thereby introduce or suppress any factor common to the two polynomials under consideration. In practice, we habitually introduce factors into dividends and suppress them in divisors.

From what precedes, we may write the following

RULE.

I. Suppress all the monomial factors in the second polynomial, and introduce such factors into the first polynomial as will make its first term divisible by the first term of the second polynomial:

II. Divide the first of the prepared polynomials by the second, continuing the division as far as possible:

III. Take the second polynomial as a new dividend, and the first remainder as a new divisor, and proceed as before; and so on, continuing the operation till a remainder is found that will exactly divide the preceding divisor. This remainder will be the greatest common divisor of the given polynomials.

EXAMPLES.

1. Find the greatest common divisor of

$$3x^3 - 13x^2 + 23x - 21 \quad \text{and} \quad 12x^3 + 2x^2 - 88x + 42.$$

Suppressing the factor 2 in the second polynomial, and introducing the factor 2 into the first, we have, for the

FIRST OPERATION,

$$\begin{array}{r|l}
6x^3 - 26x^2 + 46x - 42 & \underline{6x^3 + x^2 - 44x + 21} \\
\underline{6x^3 + x^2 - 44x + 21} & 1 \\
\text{1st rem. . } - 27x^2 + 90x - 63 &
\end{array}$$

Suppressing the factor -9 in the first remainder, and proceeding as before, we have, for the

SECOND OPERATION,

$$\begin{array}{r|l}
6x^3 + x^2 - 44x + 21 & 3x^2 - 10x + 7 \\ \cline{2-2}
\underline{6x^3 - 20x^2 + 14x \phantom{{}+21}} & 2x + 7 \\
21x^2 - 58x + 21 & \\
\underline{21x^2 - 70x + 49} & \\
\text{2d rem.} \ldots\ 12x - 28 &
\end{array}$$

Suppressing the factor 4 in the second remainder, and proceeding as before, we have, for the

THIRD OPERATION,

$$\begin{array}{r|l}
3x^2 - 10x + 7 & 3x - 7 \\ \cline{2-2}
\underline{3x^2 - 7x \phantom{{}+7}} & x - 1 \\
- 3x + 7 & \\
\underline{- 3x + 7} & \\
0 &
\end{array}$$

Hence, $3x - 7$ is the greatest common divisor sought.

2. Find the greatest common divisor of

$$6x^3 - 6x^2y + 2xy^2 - 2y^3 \quad \text{and} \quad 12x^2 - 15xy + 3y^2.$$

Suppressing the factor 3 in the second polynomial, and introducing the factor 2 into the first, we have,

$$\begin{array}{r|l}
12x^3 - 12x^2y + 4xy^2 - 4y^3 & 4x^2 - 5xy + y^2 \\ \cline{2-2}
\underline{12x^3 - 15x^2y + 3xy^2 \phantom{{}-4y^3}} & 3x, \quad 3y \\
\underline{3x^2y + xy^2 - 4y^3} & \\
12x^2y + 4xy^2 - 16y^3 & \\
\underline{12x^2y - 15xy^2 + 3y^3} & \\
\text{1st rem.} \ldots\ 19xy^2 - 19y^3 &
\end{array}$$

In this case, the first term of the first partial remainder is not divisible by the first term of the divisor: to avoid fractional coefficients, we introduce the factor 4, which cannot affect the value of the common factors, as has already been shown.

Suppressing the factor $19y^2$ in the remainder, and proceeding as before, we have,

$$\begin{array}{r|l}
4x^2-5xy+y^2 & x-y \\ \cline{2-2}
\underline{4x^2-4xy} & 4x-y \\
-\ xy+y^2 & \\
\underline{-\ xy+y^2} & \\
0 &
\end{array}$$

Hence, $x-y$ is the greatest common divisor sought.

Find the greatest common divisor of the following quantities:

3. $x^3+3x^2+4x+12$ and x^3+4x^2+4x+3. *Ans.* $x+3$.

4. x^3+x^2+x-3 and x^3+3x^2+5x+3. *Ans.* x^2+2x+3.

5. x^3+1 and x^3+mx^2+mx+1 *Ans.* $x+1$.

6. $6x^3-7ax^2-20a^2x$ and $3x^2+ax-4a^2$. *Ans.* $3x+4a$.

7. $x^4-3x^3+2x^2+x-1$ and x^3-x^2-2x+2. *Ans.* $x-1$.

8. $x^4-7x^3+8x^2+28x-48$ and $x^3-8x^2+19x-14$. *Ans.* $x-2$.

9. $4x^4+9x^3+2x^2-2x-4$ and $3x^3+5x^2-x+2$. *Ans.* $x+2$.

10. $x^5+3x^4-8x^2-9x-3$ and $x^5-2x^4-6x^3+4x^2+13x+6$. *Ans.* $(x+1)(x+1)(x+1)$.

11. $6x^5-4x^4-11x^3-3x^2-3x-1$ and $4x^4+2x^3-18x^2+3x-5$. *Ans.* $2x^3-4x^2+x-1$.

12. $20x^6-12x^5+16x^4-15x^3+14x^2-15x+4$ and $15x^4-9x^3+47x^2-21x+28$. *Ans.* $5x^2-3x+4$.

13. $5a^4b^2+2a^3b^3+ca^2-3a^2b^4+bca$, $a^5+5a^3d-a^3b^2+5a^2bd$. *Ans.* a^2+ab.

55. To find the greatest common divisor of three or more quantities, we find the greatest common divisor of the first and second; then we find the greatest common divisor of this result, and the third quantity, and so on to the last.

Find the greatest common divisor of the following:

12. $x^3 - 9x^2 + 26x - 24$, $x^3 - 10x^2 + 31x - 30$, and $x^3 - 11x^2 + 38x - 40$. *Ans.* $x - 2$.

13. $x^4 - 10x^2 + 9$, $x^4 + 10x^3 + 20x^2 - 10x - 21$, and $x^4 + 4x^3 - 22x^2 - 4x + 21$. *Ans.* $x^2 - 1$.

14. $x^4 - a^4$, $x^3 + a^3$, and $x^2 - a^2$. *Ans.* $x + a$.

15. $a^5 + b^5$, $a^4 - b^4$, and $a^3 + b^3$. *Ans.* $a + b$.

16. $3x^3 - 7x^2y + 5xy^2 - y^3$, $x^2y + 3xy^2 - 3x^3 - y^3$, and $3x^3 + 5x^2y + xy^2 - y^3$. *Ans.* $3x - y$.

VII. LEAST COMMON MULTIPLE.

56. A **Common Multiple** of two or more quantities is a quantity that each will divide without a remainder.

57. The **Least Common Multiple** of two or more quantities is the simplest quantity that can be divided by each without a remainder.

58. When the quantities can be resolved into prime factors, by inspection, it is obvious that we may find their least common multiple by the following

RULE.

Form the continued product of all the prime factors, taking each the greatest number of times it enters any one of the quantities.

EXAMPLES.

1. Find the least common multiple of the quantities:

$$x-1, \quad x^2-1, \quad \text{and} \quad x^2-2x+1.$$

Factoring, we have,

$$x-1, \quad (x-1)(x+1), \quad \text{and} \quad (x-1)(x-1);$$

hence, we have for the least common multiple,

$$(x-1)(x-1)(x+1)=x^3-x^2-x+1. \quad \textit{Ans.}$$

2. Find the least common multiple of

$$x^2-y^2, \quad x^2+2xy+y^2, \quad \text{and} \quad x^3-y^3.$$

Factoring, we have,

$$(x-y)(x+y), \quad (x+y)(x+y), \quad \text{and} \quad (x-y)(x^2+xy+y^2);$$

hence, the least common multiple is,

$$(x-y)(x+y)(x+y)(x^2+xy+y^2). \quad \textit{Ans.}$$

Find the least common multiple of the following quantities:

3. $6x$, $2(x+1)$, and $2(x+2)$. *Ans.* $6x(x^2+3x+2.)$

4. a, $a+2b$, and a^2-4b^2. *Ans.* $a(a^2-4b^2)$.

5. $x-1$, $2+2x$, and $2x^2+2$. *Ans.* $2(x^4-1)$.

6. $2x(x+1)$, $3x(x-1)$, and $x(x^2-1)$. *Ans.* $6x(x^2-1)$.

59. To deduce a general rule for finding the least common multiple of two quantities, P and P′. Denote their greatest common divisor by D, and the corresponding quotients by Q and Q′. Then we shall have,

$$P=Q\times D \quad \text{and} \quad P'=Q'\times D.$$

Hence, from the definition, the least common multiple of P and P′ is equal to Q × Q′ × D; or, since Q × D = P, the least common multiple is equal to

$$P\times Q'.$$

That is, *the least common multiple of two quantities* is equal to one of the quantities multiplied into the quotient of the other by their greatest common divisor. Hence the following

RULE.

Find the greatest common divisor of the two quantities; divide one of the quantities by it; then multiply the other quantity by the quotient.

EXAMPLES.

Find the least common multiple of the following:

1. $2x^2 - xy - 6y^2$ and $3x^2 - 8xy + 4y^2$.

Their greatest common divisor is equal to $x - 2y$; hence, their least common multiple is,

$$\frac{2x^2 - xy - 6y^2}{x - 2y} \times (3x^2 - 8xy + 4y^2) = (2x + 3y)(3x^2 - 8xy + 4y^2).$$

2. $3x^2 - 5x + 2$ and $4x^3 - 4x^2 - x + 1$.

Ans. $(3x - 2)(4x^3 - 4x^2 - x + 1)$.

3. $6x^2 - x - 1$ and $2x^2 + 3x - 2$.

Ans. $(3x + 1)(2x^2 + 3x - 2)$.

60. To find the least common multiple of several quantities. Find the least common multiple of the first and second; then of that result and the third, and so on to the last.

Find the least common multiple of the following:

4. $x^3 - 6x^2 + 11x - 6$, $x^3 - 9x^2 + 26x - 24$, and $x^3 - 8x^2 + 19x - 12$. *Ans.* $(x - 1)(x - 2)(x - 3)(x - 4)$.

5. $x^4 - 10x^2 + 9$, $x^4 + 10x^3 + 20x^2 - 10x - 21$, and $x^4 + 4x^3 - 22x^2 - 4x + 21$. *Ans.* $(x^2 - 1)(x^2 - 9)(x + 7)$.

6. $x^2 - (a + b)x + ab$, $x^2 - (b + c)x + bc$, and $x^2 - (c + a)x + ca$.

Ans. $(x - a)(x - b)(x - c)$.

7. $6(a^3 - b^3)(a - b)^3$, $9(a^4 - b^4)(a - b)^2$, and $12(a^2 - b^2)^3$.

Ans. $36(a^4 - b^4)(a^3 - b^3)(a^2 - b^2)^2$.

CHAPTER III.

ALGEBRAIC FRACTIONS.

I. DEFINITIONS AND PRINCIPLES.

61. Fractional Units.—If the unit 1 be divided into any number of equal parts, each part is a *fractional unit.* Thus, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{x}$, and $\frac{1}{a}$ are fractional units.

62. An Algebraic Fraction is an algebraic expression for one or more fractional units. Thus, $\frac{a}{b}$, $\frac{3x+y}{c}$, are algebraic fractions. The first indicates that the fractional unit $\frac{1}{b}$ is taken a times, and the second indicates that the unit $\frac{1}{c}$ is taken $(3x+y)$ times.

63. Terms.—Every fraction is composed of two parts, called *terms:* the *denominator,* written below the line, shows the number of equal parts into which the unit 1 is divided; and the *numerator,* written above, shows how many of these parts are taken. Thus, in the fraction $\frac{a}{b}$, the denominator b shows that the unit 1 is divided into b equal parts, and the numerator a shows that a of these parts are taken. The fractional unit is always equal to the *reciprocal* of the denominator.

64. An Entire Quantity is one that contains no fractional part. Thus, $a^2b + cx$ is entire.

An entire quantity may be regarded as a fraction whose denominator is 1. Thus, $a^2b + cx$ is equivalent to $\frac{a^2b + cx}{1}$.

65. A Mixed Quantity is one that contains both entire and fractional parts. Thus, $a^2b + \frac{c}{d}$ is a mixed quantity.

66. If we compare the two fractions $\frac{a}{b}$ and $\frac{aq}{b}$, we see that they have the common unit $\frac{1}{b}$, and that this unit is taken a times in the *first* and aq times in the *second;* that is,

$$\frac{aq}{b} = \frac{a}{b} \times q.$$

Hence, *multiplying the numerator of a fraction by any number is equivalent to multiplying the fraction by that number.*

67. If we compare the two fractions $\frac{a}{b}$ and $\frac{a}{bq}$, we see that the fractional unit of the first is q times as great as the fractional unit of the second, and that the number of times the fractional unit is taken is the same in both; consequently, the first fraction is equal to q times the second, that is,

$$\frac{a}{bq} = \frac{a}{b} \div q.$$

Hence, *multiplying the denominator of a fraction by any number is equivalent to dividing the fraction by that number.*

68. Since we may multiply and divide any expression by the same quantity, without altering its value, it follows (Arts. **66** and **67**) that:

Both numerator and denominator of a fraction may be multiplied by the same number, without changing the value of the fraction.

In like manner, it is evident that:

Both numerator and denominator of a fraction may be divided by the same number, without changing the value of the fraction.

69. From the preceding article we see that we may introduce any factor into both terms of a fraction, or, that we may strike out any factor that is common to both terms, without changing the value of the fraction. Hence, we may change the signs of both terms of a fraction; for, changing the sign of a quantity is equivalent to introducing the factor -1.

We may also reduce an entire quantity to a fractional form, having a given denominator, by introducing the denominator both as a factor and a divisor. Thus, the quantity c being given, to reduce it to a fractional form whose denominator is b, we have,

$$c = \frac{c}{1} = \frac{bc}{b}.$$

If the denominator is a factor of the numerator, the fraction may be reduced to an entire form by striking out the factor common to both terms. Thus, we have,

$$\frac{bc}{b} = \frac{c}{1} = c.$$

II. TRANSFORMATION OF FRACTIONS.

70. The Transformation of a Fraction, is the operation of changing its form without altering its value. The term, *transformation,* in Algebra, corresponds to the term, *reduction,* in Arithmetic.

Fractions are transformed by the aid of the principles deduced in the preceding articles.

71. First Transformation.—To reduce a fraction to its simplest form.

A fraction is in its *simplest form* when its terms have no common factor. Hence, we have for the reduction of a fraction to its simplest form, the following

RULE.

Resolve both terms into their prime factors, and then suppress all that are common to both.

When the quantities cannot be factored by inspection, we may find the greatest common divisor of both terms, and then divide both numerator and denominator by it.

EXAMPLES.

1. Reduce the fraction $\frac{3ab + 6ac}{3ad + 12a}$ to its simplest form.

Factoring and suppressing the common factor, we have,

$$\frac{3ab + 6ac}{3ad + 12a} = \frac{3a(b + 2c)}{3a(d + 4)} = \frac{b + 2c}{d + 4} \quad \textit{Ans.}$$

Reduce the following fractions to their simplest forms:

2. $\frac{6a^2b + 3ac}{9ab + 3ad}$. *Ans.* $\frac{2ab + c}{3b + d}$.

3. $\frac{25bc + 5bf}{35b^2 + 15b}$. *Ans.* $\frac{5c + f}{7b + 3}$.

4. $\frac{36a^2b + 12abf}{84ab^2}$. *Ans.* $\frac{3a + f}{7b}$

5. $\frac{12acd - 4cd^2}{12cdf + 4c^2d}$. *Ans.* $\frac{3a - d}{3f + c}$.

6. $\frac{18a^2c^2 - 3acf}{27ac^2 - 6ac^3}$. *Ans.* $\frac{6ac - f}{9c - 2c^2}$.

7. $\frac{6x^2 - 7x - 20}{4x^3 - 27x + 5}$. *Ans.* $\frac{3x + 4}{2x^2 + 5x - 1}$.

8. $\frac{x^2 + x - 2}{2x^2 - 3x + 1}$. *Ans.* $\frac{x + 2}{2x - 1}$.

9. $\dfrac{3x^3 - 22x - 15}{5x^4 - 17x^3 + 18x}$. *Ans.* $\dfrac{3x^2 + 9x + 5}{5x^3 - 2x^2 - 6x}$.

10. $\dfrac{x^{3m} + x^{2m} - 2}{x^{2m} + x^{m} - 2}$. *Ans.* $\dfrac{x^{2m} + 2x^{m} + 2}{x^{m} + 2}$.

72. Second Transformation.—To reduce a mixed quantity to a fractional form.

From what was shown (Art. 69), it follows that we may reduce the entire part to a fractional form having the same unit as the fractional part, by multiplying and dividing by the denominator of the fractional part. The two parts having then the same fractional unit, may be reduced by adding their numerators and writing the sum over the common denominator.

Hence, to reduce a mixed quantity to a fractional form, we have the following

RULE.

Multiply the entire part by the denominator of the fraction; to the product annex the numerator with its sign, and under the result write the denominator.

EXAMPLES.

1. Reduce $x - \dfrac{(a^2 - x^2)}{x}$ to the form of a fraction.

Here, $x - \dfrac{a^2 - x^2}{x} = \dfrac{x^2 - (a^2 - x^2)}{x} = \dfrac{2x^2 - a^2}{x}$.

Reduce the following quantities to fractional forms:

2. $5 + \dfrac{2x - 7}{3x}$. *Ans.* $\dfrac{17x - 7}{3x}$.

3. $1 - \dfrac{x - a - 1}{a}$. *Ans.* $\dfrac{2a - x + 1}{a}$.

4. $1 + 2x - \dfrac{x - 3}{5x}$. *Ans.* $\dfrac{10x^2 + 4x + 3}{5x}$.

5. $3x - 1 - \dfrac{x + a}{3a - 2}$. *Ans.* $\dfrac{9ax - 4a - 7x + 2}{3a - 2}$.

6. $ax + 4 - \dfrac{ax^2 - y}{x + y}$. *Ans.* $\dfrac{axy + 4x + 5y}{x + y}$.

7. $3 + \dfrac{3}{x^2 - 1}$. *Ans.* $\dfrac{3x^2}{x^2 - 1}$.

8. $a^2 - ax + x^2 + \dfrac{2x^3}{a + x}$. *Ans.* $\dfrac{a^3 + 3x^3}{a + x}$.

We shall hereafter treat mixed quantities as though they were fractional, supposing them to have been reduced to a fractional form by the preceding rule.

73. Third Transformation.—To reduce a fraction to the form of a mixed quantity.

This transformation is the converse of the preceding; hence, we have the following

RULE.

Divide the numerator by the denominator, and continue the operation as far as possible; to the entire part thus found, annex the fractional part, formed by writing the remainder over the denominator.

EXAMPLES.

Reduce the following fractions to mixed forms:

1. $\dfrac{ax + a^2}{x}$. *Ans.* $a + \dfrac{a^2}{x}$.

2. $\dfrac{ab - 2a^2}{b}$. *Ans.* $a - \dfrac{2a^2}{b}$.

3. $\dfrac{10x^2 - 5x + 3}{5x}$. *Ans.* $2x - 1 + \dfrac{3}{5x}$.

4. $\dfrac{x^2 + 6x + 12y - 5y^2}{x - 2y + 6}$. *Ans.* $x + 2y - \dfrac{y^2}{x - 2y + 6}$.

5. $\dfrac{x^2+x}{x-1}$. *Ans.* $x+2+\dfrac{2}{x-1}$.

6. $\dfrac{4ax-2x^2-a^2}{2a-x}$. *Ans.* $2x-\dfrac{a^2}{2a-x}$.

74. Fourth Transformation.—To reduce fractions having different denominators to equivalent ones having a common denominator.

Let $\dfrac{a}{b}$, $\dfrac{c}{d}$, and $\dfrac{e}{f}$ be any three fractions. If we multiply both terms of the *first* fraction by df (Art. **68**), both terms of the *second* by bf, and both terms of the *third* by bd, we have the equivalent fractions,

$$\frac{adf}{bdf}, \quad \frac{bcf}{bdf}, \quad \text{and} \quad \frac{bde}{bdf},$$

which are of the required form. And since we may treat all fractions in a similar manner, we have the following

RULE.

Multiply each numerator into all the denominators except its own, for new numerators, and all the denominators together for a common denominator.

EXAMPLES.

Reduce the following groups of fractions to equivalent ones, having a common denominator:

1. $\dfrac{a}{b}$ and $\dfrac{a+b}{c}$ *Ans.* $\dfrac{ac}{bc}$ and $\dfrac{ab+b^2}{bc}$.

2. $\dfrac{1}{2}$, $\dfrac{a^2}{3}$, and $\dfrac{a^2+x^2}{a+x}$.

Ans. $\dfrac{3a+3x}{6a+6x}$, $\dfrac{2a^3+2a^2x}{6a+6x}$, and $\dfrac{6a^2+6x^2}{6a+6x}$.

3. $\dfrac{a}{a-b}$, $\dfrac{c-b}{ax}$, and $\dfrac{b}{c}$.

Ans. $\dfrac{a^2cx}{a^2cx - abcx}$, $\dfrac{ac^2 - abc - bc^2 + cb^2}{a^2cx - abcx}$, and $\dfrac{a^2bx - ab^2x}{a^2cx - abcx}$.

75. Least Common Denominator.—The preceding rule covers all cases; but if the denominators of the fractions have any common factors, the operations may be simplified.

Take the fractions

$$\frac{13x}{6a}, \quad \frac{2bx}{4ay}, \quad \text{and} \quad \frac{5x^2}{2a^2y^2}.$$

Here $2a$ is a common factor of the denominators, and the least common multiple of the denominators is $12a^2y^2$. This is to be taken as the least common denominator of the fractions. If we multiply both terms of each fraction by the quotient of $12a^2y^2$ by the corresponding denominator, we have,

$$\frac{26axy^2}{12a^2y^2}, \quad \frac{6abxy}{12a^2y^2}, \quad \text{and} \quad \frac{30x^2}{12a^2y^2}.$$

These fractions have the least possible common denominator; and as all similar cases may be treated in the same way, we have the following

RULE.

Find the least common multiple of all the denominators for a common denominator: divide this by each denominator, separately, and multiply the quotients by the corresponding numerators for new numerators.

EXAMPLES.

Reduce the following groups of fractions to their least common denominator:

1. $\dfrac{1}{a - x}$ and $\dfrac{1}{a^2 - x^2}$. *Ans.* $\dfrac{a + x}{a^2 - x^2}$ and $\dfrac{1}{a^2 - x^2}$

2. $\dfrac{1}{a^3(a + x)}$, $\dfrac{1}{a^2(a - x)}$, and $\dfrac{2}{a^2(a^2 - x^2)}$.

Ans. $\dfrac{a - x}{a^3(a^2 - x^2)}$, $\dfrac{a(a + x)}{a^3(a^2 - x^2)}$, and $\dfrac{2a}{a^3(a^2 - x^2)}$.

3. $\frac{x}{a-1}$, $\frac{y}{a^2-1}$, and $\frac{z}{a^2-2a+1}$.

Ans. $\frac{x(a^2-1)}{(a+1)(a-1)^2}$, $\frac{y(a-1)}{(a+1)(a-1)^2}$, and $\frac{z(a+1)}{(a+1)(a-1)^2}$.

4. $\frac{x+1}{x-1}$, $\frac{x^2+1}{x^2+x+1}$, and $\frac{x^3}{x^3-1}$.

Ans. $\frac{(x+1)(x^2+x+1)}{x^3-1}$, $\frac{(x^2+1)(x-1)}{x^3-1}$, and $\frac{x^3}{x^3-1}$.

III. ADDITION OF FRACTIONS.

76. Quantities can only be added when they have the same unit. Hence, all fractions must first be reduced to equivalent ones having the same fractional unit; then the sum of the numerators will designate the number of times this unit is to be taken. We have, therefore, for the addition of fractions, the following

RULE.

Reduce the fractions to a common denominator; then add the numerators together, and place their sum over the common denominator.

EXAMPLES.

1. Find the sum of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$.

Reducing and adding, we have,

$$\frac{adf}{bdf} + \frac{cbf}{bdf} + \frac{ebd}{bdf} = \frac{adf + cbf + ebd}{bdf} \quad Ans.$$

Add the following groups:

2. $a - \frac{3x^2}{b}$ and $b + \frac{2ax}{c}$. *Ans.* $a + b + \frac{2abx - 3cx^2}{bc}$.

3. $\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$. *Ans.* $x + \frac{x}{12}$.

4. $\frac{x-2}{3}$ and $\frac{4x}{7}$. *Ans.* $\frac{19x-14}{21}$.

5. $x+\frac{x-2}{3}$ and $3x+\frac{2x-3}{4}$. *Ans.* $4x+\frac{10x-17}{12}$.

In adding mixed quantities, as in examples 2d and 5th, we may add the entire parts and the fractional parts separately, and then take the sum of the results.

6. $4x$, $\frac{4x^2}{2a}$, and $\frac{x+a}{2x}$. *Ans.* $4x+\frac{4x^3+ax+a^2}{2ax}$.

7. $\frac{a}{a+b}$ and $\frac{b}{a-b}$. *Ans.* $\frac{a^2+b^2}{a^2-b^2}$.

8. $\frac{1}{4a^3(a+x)}$, $\frac{1}{4a^3(a-x)}$, and $\frac{1}{2a^2(a^2-x^2)}$.

Ans. $\frac{1}{a^2(a^2-x^2)}$.

9. $\frac{x(a+x)}{a-x}$, $\frac{5ax-x^2}{x-a}$, and $\frac{2a^2}{a-x}$.

Ans. $2(a-x)$.

10. $\frac{a-x}{a-b}$, $\frac{c}{a+b}$, and $\frac{d}{a+x}$.

Ans. $\frac{a^3+a^2(b+c+d)-a(x^2-cx+bc)-b(x^2+cx+bd)}{a^3+a^2x-ab^2-b^2x}$.

IV. SUBTRACTION OF FRACTIONS.

77. From what precedes, it is obvious that we must first reduce the fractions to a common unit; then, the difference of their numerators, written over the common denominator, will indicate the difference of the fractions. Hence, the following

RULE.

I. Reduce the fractions to a common denominator:

II. Subtract the numerator of the subtrahend from the numerator of the minuend, and place the difference over the common denominator.

EXAMPLES.

1. From $\dfrac{x-a}{2b}$ subtract $\dfrac{2a-4x}{3c}$.

Reducing to a common denominator and indicating the subtraction, we have,

$$\frac{3cx-3ac}{6bc}-\frac{4ab-8bx}{6bc}.$$

Hence, we have for the remainder,

$$\frac{3cx-3ac-4ab+8bx}{6bc}.$$

Perform the subtractions indicated below:

2. $\dfrac{3x}{7}-\dfrac{2x}{9}$. *Ans.* $\dfrac{13x}{63}$.

3. $\dfrac{x+a}{b}-\dfrac{c}{d}$. *Ans.* $\dfrac{dx+ad-bc}{bd}$.

4. $\dfrac{3x+a}{5b}-\dfrac{2x+7}{8}$. *Ans.* $\dfrac{24x+8a-10bx-35b}{40b}$.

5. $\left(3x+\dfrac{x}{b}\right)-\left(x-\dfrac{x-a}{c}\right)$.

Ans. $2x+\dfrac{cx+bx-ab}{bc}$.

Perform the additions and subtractions indicated below:

1. $\dfrac{1}{x-1}-\left(\dfrac{1}{x+2}+\dfrac{3}{(x+2)^2}\right)$. *Ans.* $\dfrac{9}{(x-1)(x+2)^2}$.

2. $\dfrac{5}{2(x+1)}-\left(\dfrac{1}{10(x-1)}+\dfrac{24}{5(2x+3)}\right)$.

Ans. $\dfrac{2x-3}{(x^2-1)(2x+3)}$.

3. $\dfrac{3+2x}{2-x}+\dfrac{16x-x^2}{x^2-4}-\dfrac{2-3x}{2+x}$. *Ans.* $\dfrac{1}{x+2}$.

4. $\dfrac{3}{1-2x}-\dfrac{7}{1+2x}-\dfrac{4-20x}{4x^2-1}$. *Ans.* 0.

5. $\dfrac{1}{a+b}+\dfrac{b}{a^2-b^2}-\dfrac{a}{a^2+b^2}$. *Ans.* $\dfrac{2ab^2}{a^4-b^4}$.

Emanuel
McCorm

V. MULTIPLICATION OF FRACTIONS.

78. Let $\frac{a}{b}$ and $\frac{c}{d}$ represent any two fractions; and let it be required to find their product.

If we multiply $\frac{a}{b}$ by c, the product will be $\frac{ac}{b}$, obtained by multiplying the numerator by c (Art. **66**); but this product is d times too great, since we multiplied $\frac{a}{b}$ by a quantity d times too great. Hence, to obtain the true product we must divide by d, which is done (Art. **67**), by multiplying the denominator by d. We have then,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

If the numerator and denominator of the indicated product have common factors, they must be cancelled before the multiplication. Hence, we have the following

RULE.

I. Indicate the required product:

II. Cancel all factors common to the numerator and denominator, and then multiply together the remaining numerators for a new numerator, and the remaining denominators for a new denominator.

The above rule holds good for forming the continued product of any number of fractions.

EXAMPLES.

1. Multiply $a + \frac{bx}{a}$ by $\frac{c}{d}$.

First, $$a + \frac{bx}{a} = \frac{a^2 + bx}{a};$$

hence, $$\frac{a^2 + bx}{a} \times \frac{c}{d} = \frac{a^2c + bcx}{ad}.$$

Perform the multiplications indicated below:

2. $\frac{2x}{a} \times \frac{3ab}{c} \times \frac{3ac}{2b}$. *Ans.* $9ax$.

3. $\left(b + \frac{bx}{a}\right) \times \frac{a}{x}$. *Ans.* $\frac{ab + bx}{x}$.

4. $\frac{x^2 - b^2}{bc} \times \frac{x^2 + b^2}{b + c}$. *Ans.* $\frac{x^4 - b^4}{b^2c + bc^2}$.

5. $\left(x + \frac{x + 1}{a}\right) \times \frac{x - 1}{a + b}$. *Ans.* $\frac{ax^2 - ax + x^2 - 1}{a^2 + ab}$.

6. $\left(a + \frac{ax}{a - x}\right) \times \frac{a^2 - x^2}{x + x^2}$. *Ans.* $\frac{a^2 (a + x)}{x (1 + x)}$.

7. $\frac{(a - b)^2}{b + a} \times \frac{b}{x (a - b)}$. *Ans.* $\frac{(a - b) b}{x (a + b)}$.

8. $\frac{a^2 - b^2}{4} \times \frac{(a + b)}{(a - b)^2}$. *Ans.* $\frac{(a + b)^2}{4 (a - b)}$.

9. $\frac{x^2 + xy}{x^2 + y^2} \times \frac{x^3 - y^3}{xy (x + y)}$. *Ans.* $\frac{x^3 - y^3}{y (x^2 + y^2)}$.

10. $\frac{3ax}{4by} \times \frac{a^2 - x^2}{c^2 - x^2} \times \frac{bc + bx}{a^2 + ax} \times \frac{c - x}{a - x}$. *Ans.* $\frac{3x}{4y}$.

11. $\frac{1 - x^2}{1 + y} \times \frac{1 - y^2}{x + x^2} \times \left(1 + \frac{x}{1 - x}\right)$. *Ans.* $\frac{1 - y}{x}$.

12. $\frac{x (a - x)}{a^2 + 2ax + x^2} \times \frac{a (a + x)}{a^2 - 2ax + x^2}$. *Ans.* $\frac{ax}{a^2 - x^2}$.

13. $\frac{a^4 - b^4}{a^2 - 2ab + b^2} \times \frac{a - b}{a^2 + ab}$. *Ans.* $\frac{a^2 + b^2}{a}$.

14. $\left(\frac{x^2}{a^2} - \frac{x}{a} + 1\right) \times \left(\frac{x^2}{a^2} + \frac{x}{a} + 1\right)$ *Ans.* $\frac{x^4}{a^4} + \frac{x^2}{a^2} + 1$.

15. $\left(x^2 - x + 1\right) \times \left(\frac{1}{x^2} + \frac{1}{x} + 1\right)$ *Ans.* $x^2 + 1 + \frac{1}{x^2}$.

VI. DIVISION OF FRACTIONS.

79. Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions, and let it be required to divide the first by the second. The quotient may be indicated thus:

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)}.$$

If we multiply both terms of this complex fraction by $\frac{d}{c}$, which will not change the value of the fraction (Art. **68**), the new numerator will be $\frac{ad}{bc}$, and the new denominator $\frac{cd}{dc}$, which is equal to 1.

Hence,
$$\frac{a}{b} \div \frac{c}{d} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{\left(\frac{ad}{bc}\right)}{1} = \frac{ad}{bc}.$$

This last result might have been obtained by simply inverting the terms of the divisor and multiplying the dividend by the resulting fraction. Hence, for the division of fractions, we have the following

RULE.

Invert the terms of the divisor, and proceed as in multiplication of fractions (Art. **78***).*

EXAMPLES.

1. Divide, $a - \frac{b}{2c}$ by $\frac{f}{g}$.

Here, $a - \frac{b}{2c} = \frac{2ac - b}{2c}$.

Hence, $\left(a - \frac{b}{2c}\right) \div \frac{f}{g} = \frac{2ac - b}{2c} \times \frac{g}{f} = \frac{2acg - bg}{2cf}$. *Ans.*

Perform the divisions indicated below:

2. $\frac{4x^2}{7} \div 5x.$ *Ans.* $\frac{4x}{35}.$

3. $\frac{x+1}{6} \div \frac{2x}{3}.$ *Ans.* $\frac{x+1}{4x}.$

4. $\frac{x}{x-1} \div \frac{x}{2}.$ *Ans.* $\frac{2}{x-1}.$

5. $\frac{x-b}{8cd} \div \frac{3cx}{4d}.$ *Ans.* $\frac{x-b}{6c^2x}.$

6. $\frac{x^4-b^4}{x^2-2bx+b^2} \div \frac{x^2+bx}{x-b}.$ *Ans.* $x+\frac{b^2}{x}.$

7. $\frac{ax-1}{1-x} \div \frac{a}{1-x^2}.$ *Ans.* $\frac{ax(1+x)-x-1}{a}.$

8. $\frac{a+1}{a-1} \div \frac{1+a}{1-a^2}.$ *Ans.* $-(1+a).$

9. $\frac{ax-x^2}{(a+x)^2} \div \frac{x^2}{a^2-x^2}.$ *Ans.* $\frac{(a-x)^2}{x(a+x)}.$

10. $\frac{4(a^2-ab)}{b(a+b)^2} \div \frac{6ab}{a^2-b^2}.$ *Ans.* $\frac{2(a-b)^2}{3b^2(a+b)}.$

11. $\frac{2y^2}{x^3+y^3} \div \frac{y}{y+x}.$ *Ans.* $\frac{2y}{x^2-xy+y^2}.$

12. $\left(\frac{x^2}{y^3}+\frac{1}{x}\right) \div \left(\frac{x}{y^2}-\frac{1}{y}+\frac{1}{x}\right)$ *Ans.* $\frac{x+y}{y}.$

13. $\left(\frac{a}{a+b}+\frac{b}{a-b}\right) \div \left(\frac{a}{a-b}-\frac{b}{a+b}\right).$ *Ans.* 1.

14. $\left(\frac{x+2y}{x+y}+\frac{x}{y}\right) \div \left(\frac{x+2y}{y}-\frac{x}{x+y}\right).$ *Ans.* 1.

15. $\left(x^4-\frac{1}{x^4}\right) \div \left(x+\frac{1}{x}\right).$ *Ans.* $x^3-x+\frac{1}{x}-\frac{1}{x^3}.$

16. $\left(x^2+\frac{1}{x^2}+1\right) \div \left(\frac{1}{x}-1+x\right).$ *Ans.* $x+1+\frac{1}{x}.$

~~17.~~ $\left(a^2 - b^2 - c^2 - 2bc\right) \div \dfrac{a+b+c}{a+b-c}$.

Ans. $a^2 - b^2 + c^2 - 2ac$.

~~18.~~ $\left(\dfrac{x^2}{2a^2} - 4 + \dfrac{6a^2}{x^2}\right) \div \left(\dfrac{x}{2a} - \dfrac{3a}{x}\right)$. *Ans.* $\dfrac{x^2 - 2a^2}{ax}$.

GENERAL EXAMPLES.

1. Add $\dfrac{1+x^2}{1-x^2}$ to $\dfrac{1-x^2}{1+x^2}$. *Ans.* $\dfrac{2(1+x^4)}{1-x^4}$.

2. Add $\dfrac{1}{1+x}$ to $\dfrac{1}{1-x}$ *Ans.* $\dfrac{2}{1-x^2}$.

3. From $\dfrac{a+b}{a-b}$ subtract $\dfrac{a-b}{a+b}$. *Ans.* $\dfrac{4ab}{a^2-b^2}$.

4. From $\dfrac{1+x^2}{1-x^2}$ subtract $\dfrac{1-x^2}{1+x^2}$. *Ans.* $\dfrac{4x^2}{1-x^4}$.

5. Multiply $\dfrac{x^2 - 9x + 20}{x^2 - 6x}$ by $\dfrac{x^2 - 13x + 42}{x^2 - 5x}$.

Ans. $\dfrac{x^2 - 11x + 28}{x^2}$.

6. Multiply $\dfrac{x^4 - b^4}{x^2 + 2bx + b^2}$ by $\dfrac{x^2 + bx}{x - b}$. *Ans.* $x^3 + b^2x$.

7. Divide $\dfrac{a+x}{a-x} + \dfrac{a-x}{a+x}$ by $\dfrac{a+x}{a-x} - \dfrac{a-x}{a+x}$.

Ans. $\dfrac{a^2 + x^2}{2ax}$.

~~8.~~ Divide $1 + \dfrac{n-1}{n+1}$ by $1 - \dfrac{n-1}{n+1}$. *Ans.* n.

~~9.~~ $\dfrac{1+x^2}{1-x^2} + \dfrac{1-x^2}{1+x^2}$. *Ans.* $\dfrac{2(1+x^4)}{1-x^4}$.

~~10.~~ $\dfrac{1}{1+x} + \dfrac{1}{1-x}$. *Ans.* $\dfrac{2}{1-x^2}$.

~~11.~~ $\dfrac{a+b}{a-b} - \dfrac{a-b}{a+b}$. *Ans.* $\dfrac{4ab}{a^2-b^2}$.

12. $\dfrac{1+x^2}{1-x^2} - \dfrac{1-x^2}{1+x^2}$. *Ans.* $\dfrac{4x^2}{1-x^4}$

13. $\dfrac{1+x^2}{1-x^2} \div \dfrac{1-x^2}{1+x^2}$. *Ans.* $\dfrac{(1+x^2)^2}{(1-x^2)^2}$.

14. $\dfrac{x^4-b^4}{x^2-2bx+b^2} \div \dfrac{x^2+bx}{x-b}$. *Ans.* $\dfrac{x^2+b^2}{x}$.

VII. OF THE SYMBOLS 0, ∞ AND $\frac{0}{0}$.

80. The symbol 0 is called *zero.* It sometimes stands for *no quantity,* and sometimes for *a quantity less than any assignable quantity.* If we subtract a from a, we have $a - a = 0$; the result of this subtraction is *no quantity,* and the 0 that indicates this result, is the absolute 0 of arithmetic.

If in the fraction $\frac{a}{x}$ we suppose a to remain fixed in value while x increases, the value of the fraction will decrease. When x becomes exceedingly great, the value of the fraction becomes exceedingly small; and, finally, when x becomes *greater* than any assignable quantity, the corresponding value of a becomes *less* than any assignable quantity. This quantity is said to become 0, though in reality it is an *infinitesimal;* that is, it is a quantity so small that it cannot be compared with a finite quantity; and hence its numerical value is 0.

The symbol ∞ is called *infinity.* It stands for *a quantity greater than any assignable quantity.* If in the fraction $\frac{a}{x}$ we suppose a to remain fixed in value whilst x continually diminishes, the value of the fraction will increase. When x becomes exceedingly small, the value of the fraction becomes exceedingly large; and, finally, when x becomes *less* than any assignable quantity, the value of the fraction becomes *greater* than any assignable quantity, and is said to become *infinite.* This is the *infinity* of algebra, and is not to be confounded with the infinity of common language.

We say in algebra that *a finite quantity divided by zero is infinite,* and that a finite quantity *divided by infinity is zero.* The two symbols $\frac{a}{0}$ and ∞ are then equivalent; as are also the two symbols $\frac{a}{\infty}$ and 0. It will be seen that both 0 and ∞, as here explained, may be proper and legitimate answers to certain mathematical questions.

The symbol $\frac{0}{0}$ is called *the symbol of indetermination;* because it may designate a quantity which has an infinite number of values.

But the symbol does not always indicate indetermination. It may arise from the existence of a common factor in both terms of a fraction, which factor becomes 0 under a particular hypothesis. Such fractions are called *vanishing fractions.*

To illustrate, let us consider three cases:

1°. Let $$x = \frac{a^3 - b^3}{a^2 - b^2}.$$

If we suppose $a = b$, this reduces to $\frac{0}{0}$; but if we strike out the common factor $(a - b)$, we have,

$$x = \frac{a^2 + ab + b^2}{a + b}.$$

If we now suppose $a = b$, we find $x = \frac{3a}{2}$, which is the true value of x for the particular supposition.

2°. If we have $$x = \frac{a^2 - b^2}{(a - b)^2},$$

and make $a = b$, we find $x = \frac{0}{0}$; but if we strike out the factor $(a-b)$, and then make $a=b$, we find $x=\frac{2a}{0}=\infty$; this is the true value of the fraction.

3°. If we have $$x=\frac{(a-b)^2}{a^3-b^3},$$

and make $a=b$, we find $x=\frac{0}{0}$; but if we strike out the factor $(a-b)$, and then make $a=b$, we find $x=\frac{0}{3a^2}=0$, which is the true value of x.

In these cases we see that the true value of a vanishing fraction may be either *finite*, *infinite*, or *zero*.

Hence, we may have

$\frac{0}{0}$ equal to a *finite* quantity;

$\frac{0}{0}$ equal to an *infinite* quantity; or,

$\frac{0}{0}$ equal to *zero*.

Therefore, before pronouncing upon the nature of the expression, $\frac{0}{0}$, it is necessary to ascertain whether it does not arise from the existence of a common factor in both numerator and denominator, which becomes 0 under a particular hypothesis. If it does not arise from the existence of such a factor, we conclude that the expression is *indeterminate.* If it does arise from the existence of such a factor, strike it out, and then make the particular supposition.

In general, it will be found that when a fraction reduces to $\frac{0}{0}$, for a particular hypothesis, the symbol is not the true *symbol of indetermination.* But when the symbol $\frac{0}{0}$ results from *two independent* hypotheses, it is then a true symbol of indetermination (Art. **101**, 6°).

CHAPTER IV.

EQUATIONS OF THE FIRST DEGREE.

I. DEFINITIONS AND PRINCIPLES.

81. **An Equation** is an algebraic expression of equality between two quantities. Thus,

$$x = a + b,$$

is an equation; it expresses that the quantity denoted by x is equal to the sum of the quantities represented by a and b.

An equation is composed of two parts, connected by the sign of equality. The part on the left of this sign is called the *first member;* that on the right, the *second member.* The second member of an equation is often 0.

82. An equation may contain *one unknown quantity* only, or it may contain *more than one.* Equations are also classified according to degrees. *The degrees are indicated by the exponents of the unknown quantities.*

In equations containing but one unknown quantity, the degree is denoted by the greatest exponent of that quantity in any term.

In equations containing more than one unknown quantity, the degree is denoted by the greatest sum of the exponents of the unknown quantities in any term. Thus,

$$\left.\begin{array}{l} ax + b = cx + d \\ ax + 3by + cz + 8d = 0 \end{array}\right\} \text{are equations of the first degree.}$$

$$\left.\begin{array}{l} ax^2 + 2bx + c = 0 \\ ax^2 + bxy + cy^2 + d = 0 \end{array}\right\}\text{ are equations of the second degree.}$$

$$\left.\begin{array}{l} a^2x^3 + 2dgx^2 = abx - c^3 \\ 4axy^2 - 2cy^3 + abxy = 3 \end{array}\right\}\text{ are equations of the third degree,}$$

and so on.

Equations of the first degree are also called Simple Equations.

83. Numerical Equations.—Equations are distinguished as *numerical equations* and *literal equations.*

A numerical equation is one that contains no literal part, with the exception of the unknown quantity, which is always denoted by a letter. Thus,

$$4x - 3 = 2x + 5, \qquad 3x^2 - x = 8,$$

are *numerical equations.*

A Literal Equation is one in which a part, or all of the known quantities, are represented by letters. Thus,

$$bx^2 + ax - 3x = 5, \quad \text{and} \quad cx + dx^2 = c + f,$$

are *literal equations.*

84. An Identical Equation is one in which the two members are absolutely the same, or in which one member is the result of certain operations indicated in the other. In either case, the equation is true for all possible values of the unknown quantities that enter it. Thus,

$$ax + b = ax + b, \quad (x + a)^2 = x^2 + 2ax + a^2, \quad \frac{x^2 - y^2}{x + y} = x - y,$$

are *identical equations.*

85. From the nature of an equation, it must possess the following properties:

1°. The two members must be composed of quantities of the same kind.

2°. The two members must be equal to each other.

3°. The essential sign of the two members must be the same.

86. An Axiom is a self-evident proposition.

The following axioms are employed in the *transformation* and *solution* of equations:

1. If equal quantities be added to both members of an equation, the equality of the members will not be destroyed.
2. If equal quantities be subtracted from both members of an equation, the equality will not be destroyed.
3. If both members of an equation be multiplied by equal quantities, the products will be equal.
4. If both members of an equation be divided by equal quantities, the quotients will be equal.
5. Like powers of the two members of an equation are equal.
6. Like roots of the two members of an equation are equal.

II. SOLUTION OF EQUATIONS.

87. Solution.—The solution of an equation is the operation of finding such a value for the unknown quantity as, being substituted for it, will satisfy the equation—that is, make the two members equal. Every such value is called a *root* of the equation.

Verification.—The verification of an equation is the operation of proving the equality of the two members, by simply substituting the root found for the unknown quantity.

Transformation.—The transformation of an equation is any operation by which we change its form, without destroying the equality of its members.

88. First Transformation.—To reduce an equation, some of whose terms are fractional, to one in which all of the terms shall be entire.

Take the equation,

$$\frac{2x}{3} - \frac{3x}{4} + \frac{x}{6} = 11.$$

Reducing the fractions to their least common denominator, we have,

$$\frac{8x}{12} - \frac{9x}{12} + \frac{2x}{12} = 11.$$

Multiplying both members of this equation by 12, we have,

$$8x - 9x + 2x = 132.$$

Since all similar cases may be treated in the same manner, we have the following

RULE.

Form the least common multiple of all the denominators, and multiply both members of the equation by it, reducing fractional to entire terms.

This operation is called clearing of fractions.

EXAMPLES.

1. Reduce $\frac{x}{5} + \frac{x}{4} - 3 = 20$, to an equation containing only entire terms.

The least common multiple of the denominators is 20; multiplying both members by this number, and reducing, we have,

$$4x + 5x - 60 = 400.$$

In this example, we clear the equation of fractions *by multiplying each numerator into the quotient of the least common multiple of the denominators by the corresponding denominator, and omitting all the denominators.*

Clear the following equations of fractions:

2. $\frac{x}{5} + \frac{x}{7} - 4 = 3$. *Ans.* $7x + 5x - 140 = 105$.

3. $\frac{a}{b} - \frac{x}{d} + f = g.$ *Ans.* $ad - bx + bdf = bdg.$

4. $\frac{ax}{b} - \frac{2c^2x}{ab} + 4a = \frac{4bc^2x}{a^3} - \frac{5a^3}{b^2} + \frac{2c^3}{a} - 36.$

Ans. $a^4bx - 2a^2bc^2x + 4a^4b^2 = 4b^3c^2x - 5a^6 + 2a^2b^2c^3 - 36a^3b^2.$

5. $\frac{5x+8}{6} - \frac{2x-9}{3} = 4.$ *Ans.* $5x + 8 - 4x + 18 = 24.$

89. Second Transformation.—To transpose a term from one member to the other.

Let us take the equation,

$$ax + b = d - cx.$$

If we add to both members the quantity $cx - b$, and reduce the result to its simplest form, we have

$$ax + cx = d - b.$$

Here cx has been transposed from the second member to the first, and b from the first member to the second, by changing their signs. Since we may treat all like cases in a similar manner, we have for the transposition of terms the following

RULE.

Any term of an equation may be transposed from one member to the other, provided we change its sign.

90. Solution of Equations.—To deduce a rule for solving equations of the first degree containing but one unknown quantity, let us solve the equation,

$$\frac{a+b}{c}x - d = bx - \frac{a+d}{a}.$$

Clearing of fractions, we have

$$a(a+b)x - acd = abcx - c(a+d).$$

Performing the indicated multiplications, we have

$$a^2x + abx - acd = abcx - ca - cd.$$

Transposing all the terms containing x to the first member, and all the known terms to the second, we have

$$a^2x + abx - abcx = acd - ac - cd.$$

Factoring the first member, we obtain

$$(a^2 + ab - abc)x = acd - ac - cd.$$

Dividing both members by the coefficient of x, we have

$$x = \frac{acd - ac - cd}{a^2 + ab - abc}.$$

Any other equation of the first degree may be solved in a similar manner; hence, to solve an equation of the first degree, we have the following

RULE.

I. Clear the equation of fractions, and perform all the operations indicated:

II. Transpose all the terms containing the unknown quantity to the first member, and all known terms to the second member, reducing both members to their simplest form:

III. Resolve the first member into two factors, one of which is the unknown quantity; the other one will be the algebraic sum of its several coefficients:

IV. Divide both members by the coefficient of the unknown quantity; the second member of the resulting equation will be the value of the unknown quantity.

EXAMPLES.

Solve the following equations:

1. $\frac{5x}{12} - \frac{4x}{3} - 13 = \frac{7}{8} - \frac{13x}{6}.$

Clearing of fractions,

$$10x - 32x - 312 = 21 - 52x.$$

Transposing and reducing,

$$30x = 333;$$
$$x = 11.1 \quad \textit{Ans.}$$

This result may be verified by substituting 11.1 for x in the given equation. It will be found that it will make the two members equal. Every equation, after solution, may be verified.

2. $x + 18 = 3x - 5.$ *Ans.* $x = 11\frac{1}{2}.$

3. $6 - 2x + 10 = 20 - 3x - 2.$ *Ans.* $x = 2.$

4. $x + \frac{1}{2}x + \frac{1}{3}x = 11.$ *Ans.* $x = 6.$

5. $2x - \frac{1}{2}x + 1 = 5x - 2.$ *Ans.* $x = \frac{6}{7}.$

6. $3ax + \frac{a}{2} - 3 = bx - a.$ *Ans.* $x = \frac{6 - 3a}{6a - 2b}.$

7. $\frac{x-3}{2} + \frac{x}{3} = 20 - \frac{x-19}{2}.$ *Ans.* $x = 23\frac{1}{4}.$

8. $\frac{x+3}{2} + \frac{x}{3} = 4 - \frac{x-5}{4}.$ *Ans.* $x = 3\frac{6}{13}.$

9. $\frac{ax-b}{4} + \frac{a}{3} = \frac{bx}{2} - \frac{bx-a}{3}$ *Ans.* $x = \frac{3b}{3a-2b}.$

10. $\frac{3ax}{a} - \frac{2bx}{d} - 4 = f.$ *Ans.* $x = \frac{cdf + 4cd}{3ad - 2bc}.$

Tuition

11. $\frac{8ax-b}{7} - \frac{3b-c}{2} = 4 - b.$ *Ans.* $x = \frac{56 + 9b - 7c}{16a}.$

12. $\frac{x}{5} - \frac{x-2}{3} + \frac{x}{2} = \frac{13}{3}.$ *Ans.* $x = 10.$

13. $\frac{x}{a} - \frac{x}{b} + \frac{x}{c} - \frac{x}{d} = f.$

Ans. $x = \frac{abcdf}{bcd - acd + abd - abc}.$

14. $x - \frac{3x-5}{13} + \frac{4x-2}{11} = x + 1$ *Ans.* $x = 6.$

15. $\frac{x}{7} - \frac{8x}{9} - \frac{x-3}{5} = -12\frac{29}{45}$. *Ans.* $x = 14$.

16. $2x - \frac{4x-2}{5} = \frac{3x-1}{2}$. *Ans.* $x = 3$.

17. $3x + \frac{bx-d}{3} = x + a$. *Ans.* $x = \frac{3a+d}{6+b}$.

~~18.~~ $\frac{(a+b)(x-b)}{a-b} - 3a = \frac{4ab-b^2}{a+b} - 2x + \frac{a^2-bx}{b}$.

Ans. $x = \frac{a^4 + 3a^3b + 4a^2b^2 - 6ab^3 + 2b^4}{2b(2a^2 + ab - b^2)}$.

19. $x = 3x - \frac{1}{2}(4 - x) + \frac{1}{3}$. *Ans.* $x = \frac{2}{3}$.

20. $\frac{3x-7}{5} + \frac{25-4x}{9} = \frac{5x-14}{3}$. *Ans.* $x = 4$.

21. $\frac{2x+5}{13} + \frac{40-x}{8} = \frac{10x-427}{19}$. *Ans.* $x = 56$.

~~22.~~ $\frac{x}{7} - \frac{x-5}{11} + 5 = x - \left(\frac{2x}{77} + 1\right)$. *Ans.* $x = 7$.

23. $\frac{x-1}{2} + \frac{x-2}{3} = \frac{x+3}{4} + \frac{x+4}{6} + 1$. *Ans.* $x = 8\frac{3}{5}$.

24. $\frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{x-5}{x-6} - \frac{x-6}{x-7}$. *Ans.* $x = 4\frac{1}{2}$.

~~25.~~ $\left(x + \frac{5}{2}\right)\left(x - \frac{3}{2}\right) - (x+5)(x-3) + \frac{3}{4} = 0$.

Ans. $x = 12$.

26. $\frac{6x+7}{15} - \frac{2x-2}{7x-6} = \frac{2x+1}{5}$. *Ans.* $x = 3$.

~~27.~~ $\frac{1}{x-2} - \frac{1}{x-4} = \frac{1}{x-6} - \frac{1}{x-8}$. *Ans.* $x = 5$.

28. $(x+1)^2 = (5+x)x - 2$. *Ans.* $x = 1$.

29. $\frac{2}{2x-5} + \frac{1}{x-3} = \frac{6}{3x-1}$. *Ans.* $x = \frac{79}{29}$.

30. $\frac{x}{a} + \frac{x}{b-a} = \frac{a}{b+a}$. *Ans.* $x = \frac{a^2(b-a)}{b(b+a)}$.

~~31.~~ $\frac{1}{2}\left(x-\frac{a}{3}\right)-\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=0.$ *Ans.* $x=\frac{8a}{25}.$

~~32. 1.~~$2x-\frac{.18x-.05}{.5}=.4x+8.9.$ *Ans.* $x=20.$

~~33.~~ $4.8x-\frac{.72x-.05}{.5}=1.6x+8.9.$ *Ans.* $x=5.$

III. SOLUTION OF PROBLEMS.

91. **A Problem**, in Algebra, is a question proposed. Its solution requires the values of certain unknown quantities in terms of those that are given, or known.

Solution.—The solution consists of two parts:

1°. The statement of the problem; and

2°. The solution of the equation of the problem.

Statement.—The statement of a problem is the operation of expressing, in algebraic language, the relations between the known and unknown quantities of the problem. This expression is called the *equation of the problem.*

No definite rules can be laid down for stating a problem. Sometimes the enunciation of the problem furnishes the equation immediately; and sometimes it is necessary to discover, from the enunciation, new conditions from which an equation may be formed. The conditions laid down in the enunciation of the problem are called *explicit;* those deduced from them are called *implicit.*

In most cases we may find the equation of a problem by the following

RULE.

Denote the unknown quantity by one of the final letters of the alphabet, and then indicate, by algebraic signs, the same operations on the known and unknown quantities, as would be necessary to verify the value of the unknown quantity, were it known.

PROBLEMS.

1. Find a number such, that the sum of one half, one third, and one fourth of it, augmented by 45, shall be equal to 448.

Let the required number be denoted by . . . . x.

Then, one half of it will be denoted by . . . . $\frac{x}{2}$,

one third of it . . . . by . . . . $\frac{x}{3}$,

one fourth of it . . . . . . by . . . . $\frac{x}{4}$,

and by the conditions, $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + 45 = 448.$

This is the equation of the problem, and it may be solved as already explained. Solving it, we find

$$x = 372.$$

Hence, the required number is 372.

This number may be verified as follows:

$$\frac{372}{2} + \frac{372}{3} + \frac{372}{4} + 45 = 448.$$

2. What number is it whose third part exceeds its fourth part by 16?

Let the number be denoted by . . . . x.

Then will its third part be denoted by . . . . $\frac{x}{3}$,

and its fourth part . . . . by . . . . $\frac{x}{4}$.

Hence, from the condition of the problem:

$$\frac{x}{3} - \frac{x}{4} = 16.$$

Solving, we have, $x = 192$. Hence, the required number is 192.

3. Out of a cask of wine which had leaked away a third part, 21 gallons were afterward drawn, and the cask was then half full: how much did it hold?

Let the number of gallons originally in the cask be denoted by x;

Then will the number of gallons that leaked out be denoted by $\frac{x}{3}$.

From the conditions of the problem, we have,

$$\frac{x}{3} + 21 = \frac{1}{2}x; \qquad \therefore \qquad x = 126.$$

That is, the cask held 126 gallons.

4. A fish was caught whose tail weighed 9 lbs.; his head weighed as much as his tail and half his body; his body weighed as much as his head and tail together: what was the weight of the fish?

Let x denote the weight of the body, in pounds;

Then, from the conditions of the problem, we have,

$$9 + \left(9 + \frac{x}{2}\right) = x; \qquad \therefore \qquad x = 36.$$

That is, the fish weighed 72 pounds.

5. A person engaged a workman for 48 days. For each day that he labored, he received 24 cents; and for each day that he was idle, he paid 12 cents for his board. At the end of the 48 days the account was settled, when the laborer received 504 cents. *Required the number of working days, and the number of days he was idle.*

Let the number of working days be denoted by x;
then will the number of idle days be denoted by $48 - x$;
the number of cents he earned will be $24x$;
the number of cents he paid for board will be $12(48 - x)$.
From the conditions of the problem, we have,

$$24x - 12(48 - x) = 504 \qquad \therefore \qquad x = 30.$$

That is, he wrought 30 days and was idle 18 days.

6. A fox, pursued by a greyhound, has a start of 60 leaps. He makes 9 leaps while the greyhound makes but 6; but 3 leaps of the greyhound are equivalent to 7 leaps of the fox. How many leaps must the greyhound make to overtake the fox?

Let us take one of the fox leaps as the unit of distance; then, 3 leaps of the greyhound being equal to 7 leaps of the fox, one of the greyhound leaps will be equal to $\frac{7}{3}$

Let x denote the number of leaps the greyhound must make before overtaking the fox;

Then, since the fox makes 9 leaps while the hound makes 6,

$$\frac{9}{6}x \quad \text{or} \quad \frac{3}{2}x$$

will denote the number of leaps the fox makes in the same time. Therefore,

$\frac{7}{3}x$ will denote the distance passed over by the hound; and

$\frac{3}{2}x$ the whole distance passed over by the fox.

Then, from the conditions of the problem, we have

$$\frac{7}{3}x = 60 + \frac{3}{2}x. \qquad \therefore \qquad x = 72.$$

That is, the greyhound must make 72 leaps. The fox makes 108 leaps.

7. A can do a piece of work alone in 10 days, and B in 13 days: in what time can they do it if they work together?

Denote the number of days by x, and the work to be done by 1. Then, in

1 day A can do $\frac{1}{10}$ of the work; and in

1 day B can do $\frac{1}{13}$ of the work; hence, in

x days A can do $\frac{x}{10}$ of the work; and in

x days B can do $\frac{x}{13}$ of the work.

From the conditions of the problem, we have

$$\frac{x}{10} + \frac{x}{13} = 1, \qquad \therefore \qquad x = 5\tfrac{15}{23}.$$

That is, it will take them both $5\frac{15}{23}$ days to do the work.

8. Divide \$1000 between A, B, and C, so that A shall have \$72 more than B, and C \$100 more than A.

Ans. A's share, \$324; B's, \$252; C's, \$424.

9. A and B play together at cards. A sits down with \$84 and B with \$48. Each loses and wins in turn, when it appears that A has five times as much as B. How much did A win? *Ans.* \$26.

10. A person dying, leaves half of his property to his wife, one sixth to each of two daughters, one twelfth to a servant, and the remaining \$600 to the poor: what was the amount of his property? *Ans.* \$7200.

11. A father leaves his property, amounting to \$2520, to four sons, A, B, C, and D. C is to have \$360, B as much as C and D together, and A twice as much as B less \$1000: how much do A, B, and D receive?

Ans. A, \$760; B, \$880; D, \$520.

12. An estate of \$7500 is to be divided between a widow, two sons, and three daughters, so that each son shall receive twice as much as each daughter, and the widow herself \$500 more than all the children: what was her share, and what the share of each child?

Ans. Widow's share, \$4000.
Each son, \$1000.
Each daughter, \$500.

13. A company of 180 persons consists of men, women, and children. The men are 8 more in number than the

women, and the children 20 more than the men and women together: how many of each sort in the company?

Ans. 44 men, 36 women, 100 children.

14. A father divides \$2000 among five sons, so that each elder should receive \$40 more than his next younger brother: what is the share of the youngest? *Ans.* \$320.

15. A purse of \$2850 is to be divided among three persons, A, B, and C; A's share is to be $\frac{6}{11}$ of B's share, and C is to have \$300 more than A and B together: what is each one's share? *Ans.* A's, \$450; B's, \$825; C's, \$1575.

16. Two pedestrians start from the same point; the first steps twice as far as the second, but the second makes 5 steps while the first makes but one. At the end of a certain time they are 300 feet apart. Now, allowing each of the longer paces to be 3 feet, how far will each have traveled?

Ans. 1st, 200 feet; 2d, 500 feet.

17. Two carpenters, 24 journeymen, and 8 apprentices, received at the end of a certain time \$144. The carpenters received \$1 per day, each journeyman half a dollar, and each apprentice 25 cents: how many days were they employed?

Ans. 9 days.

18. A capitalist receives a yearly income of \$2940: four fifths of his money bears an interest of 4 per cent., and the remainder of 5 per cent.: how much has he at interest?

Ans. \$70000.

19. A cistern containing 60 gallons of water has three unequal cocks for discharging it; the largest will empty it in one hour, the second in two hours, and the third in three hours: in what time will the cistern be emptied if they all run together? *Ans.* $32\frac{8}{11}$ minutes.

20. In a certain orchard $\frac{1}{2}$ are apple-trees, $\frac{1}{4}$ peach-trees, $\frac{1}{6}$ plum-trees, 120 cherry-trees, and 80 pear-trees: how many trees in the orchard? *Ans.* 2400.

21. A farmer being asked how many sheep he had, answered that he had them in five fields; in the 1st, he had $\frac{1}{4}$; in the 2d, $\frac{1}{6}$; in the 3d, $\frac{1}{8}$; in the 4th, $\frac{1}{12}$; and in the 5th, 450: how many had he? *Ans.* 1200.

22. My horse and saddle together are worth $132, and the horse is worth ten times as much as the saddle: what is the value of the horse? *Ans.* $120.

23. The rent of an estate is this year 8 per cent. greater than it was last year. This year it is $1890: what was it last year? *Ans.* $1750.

24. What number is that from which, if 5 be subtracted, $\frac{2}{3}$ of the remainder will be 40? *Ans.* 65.

25. A post is $\frac{1}{3}$ in the mud, $\frac{1}{4}$ in the water, and 10 feet above the water: what is the whole length of the post? *Ans.* 24 feet.

26. After paying $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had 66 guineas left in my purse: how many guineas were in it at first? *Ans.* 120.

27. A person was desirous of giving 3 pence apiece to some beggars, but found he had not money enough in his pocket by 8 pence; he therefore gave them each two pence and had 3 pence remaining: required the number of beggars. *Ans.* 11.

28. A person in play lost $\frac{1}{4}$ of his money, and then won 3 shillings; after which he lost $\frac{1}{3}$ of what he then had; and this done, found that he had but 12 shillings remaining: what had he at first? *Ans.* 20*s.*

29. Two persons, A and B, lay out equal sums of money in trade; A gains $126, and B loses $87, and A's money is now double B's: what did each lay out? *Ans.* $300.

30. A person goes to a tavern with a certain sum of money in his pocket, where he spends 2 shillings; he then

828[illegible]24

borrows as much money as he had left, and going to another tavern he there spends 2 shillings also; then borrowing again as much money as was left, he went to a third tavern, where, likewise, he spent 2 shillings, and borrowed as much as he had left; and again spending 2 shillings at a fourth tavern, he then had nothing remaining. What had he at first? *Ans.* 3*s.* 9*d.*

31. A farmer bought a basket of eggs, and offered them at 7 cents a dozen. But before he sold any, 5 dozen were broken by a careless boy, for which he was paid. He then sold the remainder at 8 cents a dozen, and received as much as he would have got for the whole at the first price. How many eggs had he in his basket? *Ans.* 40 dozen.

32. A cask A, contains a mixture of 12 gallons of wine and 18 gallons of water; another cask B contains a mixture of 9 gallons of wine and 3 gallons of water. How many gallons must be drawn from each to produce a mixture of 7 gallons of wine and 7 gallons of water?

Ans. From A, 10 gals.; from B, 4 gals.

33. At what time between 1 and 2 o'clock is the minute hand of a clock just 1 minute space ahead of the hour hand? *Ans.* At 1*h.* $6\frac{6}{11}$*m.*

34. A person having a hours at his disposal, how far can he ride in a coach that travels b miles per hour, and return on foot at the rate of c miles per hour?

Ans. $\dfrac{abc}{b+c}$ miles.

35. A can do a piece of work in one half the time that B can; and B can do it in two thirds the time that C can; all together can do it in six days. How many days would it take each to do it singly?

Ans. A in 11 days, B in 22 days, and C in 33 days.

IV. SIMULTANEOUS EQUATIONS.

92. **Simultaneous Equations** are those in which the values of the unknown quantities are the same in each.

If we have a single equation containing two unknown quantities, we may find an expression for one of them in terms of the other. Thus, from the equation,

$$x + 2y = 4 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

we may deduce the equation,

$$x = 4 - 2y;$$

but the value of x will not be known, unless we know the value of y. Hence, a single equation containing two unknown quantities is not sufficient to determine the value of either.

If we have a second equation, as

$$x + 3y = 5 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

we may, as before, find

$$x = 5 - 3y.$$

If we place the value of x, before found, equal to this value, we have,

$$4 - 2y = 5 - 3y,$$

from which we find, $y = 1$.

This value of y, substituted in either of the given equations, gives $x = 2$.

Hence, $x = 2$ and $y = 1$ satisfy both equations.

In order to find the values of two unknown quantities, we must therefore have two independent equations.

Since equations (1) and (2) are satisfied by the same values of x and y, they are *simultaneous*.

It has been shown that two equations are necessary to determine the values of two unknown quantities; in like manner, it may be shown that three equations are necessary to determine three unknown quantities; and so on. The *number of equations* must always be equal to the *number of unknown quantities.*

A **Group** is a collection of simultaneous equations, from which the values of the unknown quantities may be found.

V. ELIMINATION.

93. Elimination is the operation of combining groups of simultaneous equations in such a manner as to cause one of the unknown quantities to disappear; and finally, of finding an equation containing but a single unknown quantity.

There are three principal methods of elimination:

1°. *By addition or subtraction;* 2°. *By substitution;* and 3°. *By comparison.*

94. Elimination by Addition or Subtraction.—Take the equations,

$$4x - 5y = 5 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

$$3x + 2y = 21 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Multiplying both members of (1) by 2, and both members of (2) by 5, we have,

$$8x - 10y = 10 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

$$15x + 10y = 105 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

Adding (3) and (4), member to member, we have,

$$23x = 115.$$

Here, y has been eliminated *by addition.*

Take the equations,

$$2x + 4y = 26 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

$$x + 6y = 20 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

Multiplying both members of (5) by 6, and both members of (6) by 4, we have,

$$12x + 24y = 156 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

$$4x + 24y = 80 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

Subtracting (8) from (7), member from member, we have,

$$8x = 76.$$

Here, y has been eliminated *by subtraction.*

In like manner, an unknown quantity may be eliminated from any two equations. Hence, the following

RULE.

Prepare the equations so that the coefficients of the quantity to be eliminated shall be numerically equal in both; then, if the coefficients have contrary signs, add the equations, member to member; if they have the same sign, subtract them member from member.

In the above examples we *prepared* the equations by multiplying both members of each by the coefficient of the quantity to be eliminated in the other. The resulting equations will generally be simpler, if we find the least common multiple of the coefficients of the quantity to be eliminated, and then multiply both members of each equation into the quotient of this multiple by the coefficient of the quantity to be eliminated in that equation. Thus, in the last example, the least common multiple of the coefficients of y in equations (5) and (6) is 12. Multiplying both members of equation (5) by $\frac{12}{4}$, or 3, and both members of equation (6) by $\frac{12}{6}$, or 2, we have,

$$6x + 12y = 78,$$
$$2x + 12y = 40.$$

Whence, by subtraction,

$$4x = 38.$$

95. Elimination by Substitution.—Take the equations,

$$5x + 7y = 43 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$
$$11x + 9y = 69 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

From (1) we find the value of x in terms of y;

$$x = \frac{43 - 7y}{5};$$

substituting this for x, in equation (2), we have,

$$\frac{11(43 - 7y)}{5} + 9y = 69 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Here x has been eliminated *by substitution.*

In a similar manner one unknown quantity may be eliminated from any two equations. Hence, the following

RULE.

Find from one of the equations the value of the quantity to be eliminated in terms of the other; substitute this value for that quantity in the other equation.

96. Elimination by Comparison.—Take the equations,

$$5x + 7y = 43,$$
$$11x + 9y = 69.$$

Finding the value of x in terms of y, from both equations, we have,

$$x = \frac{43 - 7y}{5},$$

$$x = \frac{69 - 9y}{11}.$$

Placing these values equal to each other, we have,

$$\frac{43 - 7y}{5} = \frac{69 - 9y}{11}.$$

Here x has been eliminated *by comparison.*

In like manner one unknown quantity may be eliminated 'om any two equations. Hence, the following

RULE.

Find the value of the quantity to be eliminated in terms of the others from each equation; then place their values equal to each other.

Either of these methods may be used, according to circumstances. In some cases one will be most simple; in other cases another may be preferable. As a rule, the method by addition or subtraction is most commonly employed.

97. Applications.—To apply the methods of elimination in the solution of groups of simultaneous equations, let us take the following group of three equations:

$$5x - 6y + 4z = 15 \quad \ldots\ldots \quad (1)$$

$$7x + 4y - 3z = 19 \quad \ldots\ldots \quad (2)$$

$$2x + y + 6z = 46 \quad \ldots\ldots \quad (3)$$

Combining (1) and (2), eliminating z, we have,

$$43x - 2y = 121 \quad \ldots\ldots \quad (4)$$

Combining (2) and (3), eliminating z, we have,

$$16x + 9y = 84 \quad \ldots\ldots \quad (5)$$

Equations (4) and (5) constitute a new group of two equations containing two unknown quantities.

Combining (4) and (5), eliminating y, we have,

$$419x = 1257; \qquad \therefore \qquad x = 3.$$

Substituting $x = 3$, in either (4) or (5), we find $y = 4$. Substituting $x = 3$ and $y = 4$, in either (1), (2), or (3), we find $z = 6$. Hence, $x = 3$, $y = 4$, and $z = 6$ are the required values of the unknown quantities in the given group.

In like manner any group of simultaneous equations may be solved. Hence, the following

RULE.

I. Combine one equation of the group with each of the others, eliminating one unknown quantity: there will result a new group containing one equation less than the original group:

II. Combine one equation of this group with each of the others, eliminating a second unknown quantity: there will result a new group containing two equations less than the original group:

III. Continue the operation until a single equation is found, containing but one unknown quantity:

IV. Find the value of this unknown quantity by the preceding rules; substitute this in either one of the group of two equations, and find the value of a second unknown quantity; substitute these in either of the group of three, finding a third unknown quantity; and so on, till the values of all are found.

In making the combinations, care should be taken to make them in such a way as to obtain as simple equations as possible. When any unknown quantity does not enter all of the equations, it will in general be best to eliminate that quantity first.

EXAMPLES.

1. Given $2x + 3y = 16$, and $3x - 2y = 11$ to find the values of x and y. *Ans.* $x = 5$, $y = 2$.

2. Given $\frac{2x}{5} + \frac{3y}{4} = \frac{9}{20}$, and $\frac{3x}{4} + \frac{2y}{5} = \frac{61}{120}$ to find the values of x and y. *Ans.* $x = \frac{1}{2}$, $y = \frac{1}{3}$.

3. Given $\frac{x}{7} + 7y = 99$, and $\frac{y}{7} + 7x = 51$ to find the values of x and y. *Ans.* $x = 7$, $y = 14$.

4. Given $\frac{x}{2} - 12 = \frac{y}{4} + 8$, and $\frac{x+y}{5} + \frac{x}{3} - 8 = \frac{2y-x}{4} + 27$ to find the values of x and y. *Ans.* $x = 60$, $y =$ [illegible].

$$5.\ \text{Given}\ \left\{\begin{aligned} x+y+z&=29\\ x+2y+3z&=62\\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z&=10 \end{aligned}\right\}\ \text{to find } x,\ y, \text{ and } z.$$

Ans. $x=8,\quad y=9,\quad z=12.$

$$6.\ \text{Given}\ \left\{\begin{aligned} 2x+4y-3z&=22\\ 4x-2y+5z&=18\\ 6x+7y-z&=63 \end{aligned}\right\}\ \text{to find } x,\ y, \text{ and } z.$$

Ans. $x=3,\quad y=7,\quad z=4.$

$$7.\ \text{Given}\ \left\{\begin{aligned} x+\frac{1}{2}y+\frac{1}{3}z&=32\\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z&=15\\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z&=12 \end{aligned}\right\}\ \text{to find } x,\ y, \text{ and } z.$$

Ans. $x=12,\quad y=20,\quad z=30.$

$$8.\ \text{Given}\ \left\{\begin{aligned} 7x-2z+3u&=17\\ 4y-2z+t&=11\\ 5y-3x-2u&=8\\ 4y-3u+2t&=9\\ 3z+8u&=33 \end{aligned}\right\}\ \text{to find } x,\ y,\ z,\ u, \text{ and } t.$$

Ans. $x=2,\quad y=4,\quad z=3,\quad u=3,\quad t=1.$

Solve the following groups of simultaneous equations:

$$9.\ \text{Given}\ \left\{\begin{aligned} 3x+2y-4z&=15\\ 5x-3y+2z&=28\\ 3y+4z-x&=24. \end{aligned}\right\}\qquad \textit{Ans.}\ \left\{\begin{aligned} x&=7;\\ y&=5;\\ z&=4. \end{aligned}\right.$$

$$10.\ \text{Given}\ \left\{\begin{aligned} \frac{1}{x}+\frac{1}{y}&=1\\ \frac{1}{x}+\frac{1}{z}&=2\\ \frac{1}{y}+\frac{1}{z}&=\frac{3}{2}. \end{aligned}\right\}\qquad \textit{Ans.}\ \left\{\begin{aligned} x&=\frac{4}{3};\\ y&=4;\\ z&=\frac{4}{5}. \end{aligned}\right.$$

11. Given $\left\{\begin{array}{l} \dfrac{2}{x}+\dfrac{1}{y}=\dfrac{3}{z} \\ \dfrac{3}{z}-\dfrac{2}{y}=2 \\ \dfrac{1}{x}+\dfrac{1}{z}=\dfrac{4}{3}. \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=\dfrac{7}{6}; \\ y=-\dfrac{7}{2}; \\ z=\dfrac{21}{10}. \end{array}\right.$

12. Given $\left\{\begin{array}{l} \dfrac{3y-1}{4}=\dfrac{6z}{5}-\dfrac{x}{2}+\dfrac{9}{5} \\ \dfrac{5x}{4}+\dfrac{4z}{3}=y+\dfrac{5}{6} \\ \dfrac{3x+1}{7}-\dfrac{z}{14}+\dfrac{1}{6}=\dfrac{2z}{21}+\dfrac{y}{3}. \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=2; \\ y=3; \\ z=1. \end{array}\right.$

13. Given $\left\{\begin{array}{l} \dfrac{x}{a}+\dfrac{y}{b}=1 \\ \dfrac{x}{a}+\dfrac{z}{c}=1 \\ \dfrac{y}{b}+\dfrac{z}{c}=1. \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=\dfrac{a}{2}; \\ y=\dfrac{b}{2}; \\ z=\dfrac{c}{2}. \end{array}\right.$

14. Given $\left\{\begin{array}{r} 7x-3y=1 \\ 11z-7u=1 \\ 4z-7y=1 \\ 19x-3u=1. \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=4; \\ y=9; \\ z=16; \\ u=25. \end{array}\right.$

15. Given $\left\{\begin{array}{l} \dfrac{3x-5y}{2}+3=\dfrac{2x+y}{5} \\ 8-\dfrac{x-2y}{4}=\dfrac{x}{2}+\dfrac{y}{3}. \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=12; \\ y=6. \end{array}\right.$

16. Given $\left\{\begin{array}{l} \dfrac{3x}{10}-\dfrac{y}{15}-\dfrac{4}{9}=\dfrac{x}{12}-\dfrac{y}{18} \\ 2x-\dfrac{8}{3}=\dfrac{x}{12}-\dfrac{y}{15}+\dfrac{11}{10}. \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=2; \\ y=-1. \end{array}\right.$

VI. PRACTICAL PROBLEMS.

1. What fraction is that, to the numerator of which, if 1 be added, its value will be one third; but if 1 be added to its denominator, its value will be one fourth?

Let x denote the numerator, and
y the denominator.

From the conditions of the problem,

$$\frac{x+1}{y}=\frac{1}{3},$$
$$\frac{x}{y+1}=\frac{1}{4}.$$

Solving, we have $x=4$ and $y=15$. Hence, the required fraction is $\frac{4}{15}$.

2. Find two numbers whose sum is a and whose difference is b.

Let x denote the greater, and
y the less number.

From the conditions of the problem, we have,

$$x+y=a,$$
$$x-y=b.$$

Solving, we have, $x=\frac{a+b}{2}$, the greater, and,

$y=\frac{a-b}{2}$, the less number.

3. A person possessed a capital of 30,000 dollars, for which he drew a certain interest per annum; but he owed the sum of 20,000 dollars, for which he paid a certain interest. The interest that he received exceeded that which he paid by 800 dollars. Another person possessed 35,000 dollars, for which he received interest at the second of the above rates; but he

owed 24,000 dollars, for which he paid interest at the first of the above rates. The interest that he received exceeded that which he paid by 310 dollars. Required the two rates of interest.

Let x denote the first rate, and
y the second rate.

Then, the interest on \$30,000 at x per cent. for one year will be

$$\frac{\$30{,}000x}{100} \quad \text{or} \quad \$300x.$$

The interest on \$20,000 at y per cent. for one year will be

$$\frac{\$20{,}000y}{100} \quad \text{or} \quad \$200y.$$

From the conditions of the problem, we have,

$$300x - 200y = 800$$
$$35y - 24x = 31.$$

Solving, we have,

$$y = 5 \quad \text{and} \quad x = 6.$$

Hence, the two rates are 5 per cent. and 6 per cent.

4. There are three ingots formed by mixing three metals in different proportions.

One pound of the first contains 7 ounces of silver, 3 ounces of copper, and 6 ounces of pewter;

One pound of the second contains 12 ounces of silver, 3 ounces of copper, and 1 ounce of pewter;

One pound of the third contains 4 ounces of silver, 7 ounces of copper, and 5 ounces of pewter.

It is required to form from these three, 1 pound of a fourth ingot which shall contain 8 ounces of silver, $3\frac{3}{4}$ ounces of copper, and $4\frac{1}{4}$ ounces of pewter.

Let x denote the number of ounces taken from the first;
y the number of ounces taken from the second;
z the number of ounces taken from the third.

Now, since 1 pound or 16 ounces of the first ingot contains 7 ounces of silver, one ounce will contain $\frac{1}{16}$ of 7 ounces: that is, $\frac{7}{16}$ ounces; and

x ounces will contain $\frac{7x}{16}$ ounces of silver,

y ounces will contain $\frac{12y}{16}$ ounces of silver,

z ounces will contain $\frac{4z}{16}$ ounces of silver.

But since 1 pound of the new ingot is to contain 8 ounces of silver, we have,

$$\frac{7x}{16}+\frac{12y}{16}+\frac{4z}{16}=8 \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

In like manner, we find the two equations,

$$\frac{3x}{16}+\frac{3y}{16}+\frac{7z}{16}=3\tfrac{3}{4} \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

$$\frac{6x}{16}+\frac{y}{16}+\frac{5z}{16}=4\tfrac{1}{4} \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

Solving (1), (2), and (3), we have,

$x=8$, the number of ounces taken from the first.
$y=5$ " " " " " " second.
$z=3$ " " " " " " third.

5. What two numbers are they, whose sum is 33 and whose difference is 7? *Ans.* 20 and 13.

6. Divide the number 75 into such parts, that three times the greater may exceed seven times the less by 15.

Ans. 54 and 21.

7. In a mixture of wine and cider, ½ of the whole plus 25 gallons was wine, and ⅓ part minus 5 gallons was cider: how many gallons were there of each?

Ans. 85 of wine, and 35 of cider.

8. A bill of £120 was paid in guineas and moidores, and the number of pieces of both sorts that were used was just 100; if the guinea were estimated at 21*s.*, and the moidore at 27*s.*, how many were there of each? *Ans.* 50.

9. Two travelers set out at the same time from London and York, whose distance apart is 150 miles; they travel toward each other; one of them goes 8 miles a day, and the other 7; in what time will they meet? *Ans.* In 10 days.

10. At a certain election, 375 persons voted for two candidates, and the candidate chosen had a majority of 91: how many voted for each?

Ans. 233 for one, and 142 for the other.

11. A's age is double B's, and B's is triple C's, and the sum of all their ages is 140: what is the age of each?

Ans. A's = 84, B's = 42, and C's = 14.

12. A person bought a chaise, horse, and harness, for £60; the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness: what did he give for each?

Ans. £13 6*s.* 8*d.* for the horse.
£6 13*s.* 4*d.* for the harness.
£40 for the chaise.

13. A person has two horses, and a saddle worth £50; now, if the saddle be put on the back of the first horse, it will make his value double that of the second; but if it be put on the back of the second, it will make his value triple that of the first: what is the value of each horse?

Ans. One £30, and the other £40.

14. Two persons, A and B, have each the same income. A saves $\frac{1}{5}$ of his yearly; but B, by spending £50 per annum more than A, at the end of four years finds himself £100 in debt: what is the income of each? *Ans.* £125.

15. To divide the number 36 into three such parts that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third may be all equal to each other. *Ans.* 8, 12, and 16.

16. A footman agreed to serve his master for £8 a year and a livery, but was turned away at the end of 7 months, and received only £2 13*s.* 4*d.* and his livery: what was its value? *Ans.* £4 16*s.*

17. To divide the number 90 into four such parts that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient, so obtained, will be all equal to each other.

Ans. The parts are 18, 22, 10, and 40.

18. The hour and minute hands of a clock are exactly together at 12 o'clock: when are they next together?

Ans. 1 h. $5\frac{5}{11}$ min.

19. A man and his wife usually drank out a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days: how many days would the man be in drinking it alone? *Ans.* 20 days.

20. If A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C in 10 days, how many days would it take each person to perform the same work alone?

Ans. A, $14\frac{34}{49}$ days; B, $17\frac{23}{41}$ days; and C, $23\frac{7}{31}$ days.

21. A laborer can do a certain work expressed by a in a time expressed by b; a second laborer, the work c in a time d; a third, the work e in a time f. Required the time it would take the three laborers, working together, to perform the work g. *Ans.* $\frac{bdfg}{adf + bcf + bde}$.

22. If 32 pounds of sea water contains 1 pound of salt, how much fresh water must be added to those 32 pounds in order that the quantity of salt contained in 32 pounds of the new mixture shall be reduced to 2 ounces, or $\frac{1}{8}$ of a pound? *Ans.* 224 lbs.

23. A number is expressed by three figures; the sum of these figures is 11; the figure in the place of units is double that in the place of hundreds; and when 297 is added to this number, the sum obtained is expressed by the figures of this number reversed. What is the number? *Ans.* 326.

24. A person who possessed \$100,000, placed the greater part of it out at 5 per cent. interest, and the other part at 4 per cent. The interest which he received for the whole amounted to \$4640. Required the two parts.

Ans. \$64,000 and \$36,000.

25. A person possessed a certain capital, which he placed out at a certain interest. Another person possessed \$10,000 more than the first, and putting out his capital 1 per cent. more advantageously, had an income greater by \$800. A third possessed \$15,000 more than the first, and putting out his capital 2 per cent. more advantageously, had an income greater by \$1500. Required the capitals, and the three rates of interest.

Sums at interest,	\$30,000,	\$40,000,	\$45,000.	
Rates of interest,	4	5	6	per cent.

26. A cistern may be filled by three pipes, A, B, C. By the two first it can be filled in 70 minutes; by the first and third it can be filled in 84 minutes; and by the second and third in 140 minutes. What time will each pipe take to fill it? What time will be required, if the three pipes run together?

$$\textit{Ans.}\ \begin{cases}\text{A in 105 minutes.}\\ \text{B in 210 minutes.}\\ \text{C in 420 minutes.}\end{cases}$$

All will fill it in one hour.

27. A has 3 purses, each containing a certain sum of money. If \$20 be taken out of the first and put into the second, it will contain four times as much as remains in the first. If \$60 be taken from the second and put into the third, then this will contain $1\frac{3}{4}$ times as much as there remains in the second. Again, if \$40 be taken from the third and put into the first, then the third will contain $2\frac{7}{8}$ times as much as the first. What were the contents of each purse?

Ans. 1st. \$120.
2d. \$380.
3d. \$500.

28. A banker has two kinds of money; it takes a pieces of the first to make a crown, and b of the second to make the same sum. Some one offers him a crown for c pieces. How many of each kind must the banker give him?

Ans. 1st kind, $\frac{a(c-b)}{a-b}$; 2d kind, $\frac{b(a-c)}{a-b}$.

29. Find what each of three persons, A, B, C, is worth, knowing, 1st, that what A is worth added to l times what B and C are worth, is equal to p; 2d, that what B is worth added to m times what A and C are worth, is equal to q; 3d, that what C is worth added to n times what A and B are worth, is equal to r.

If we denote by s what A, B, and C are worth, we introduce an auxiliary unknown quantity, and resolve the question in a very simple manner.

30. Find the values of the estates of six persons, A, B, C, D, E, F, from the following conditions: 1st. The sum of the estates of A and B is equal to a; that of C and D is equal to b; and that of E and F is equal to c. 2d. The estate of A is worth m times that of C; the estate of D is worth n times that of E, and the estate of F is worth p times that of B.

This problem may be solved by means of a single equation, involving but one unknown quantity.

VII. INDETERMINATE EQUATIONS.

98. An Indeterminate Equation is one that may be satisfied by an infinite number of sets of values of the unknown quantities that enter it.

Every equation containing two unknown quantities is indeterminate.

For, let us take the equation,

$$5x - 3y = 12.$$

Making y successively equal to

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad \text{etc.},$$

we find from the equation the corresponding values of x,

$$3, \quad \frac{18}{5}, \quad \frac{21}{5}, \quad \frac{24}{5}, \quad \frac{27}{5}, \quad 6, \quad \text{etc.}$$

Now, any two corresponding values of x and y, when substituted in the given equation, will satisfy it; that is, the given equation may be satisfied by an infinite number of sets of values of x and y. Hence, it is indeterminate.

Every equation containing more than one unknown quantity is indeterminate.

For, we may assume values at pleasure for all the unknown quantities except one, and the value of this one may then be found from the equation. Hence, the equation may be satisfied by an infinite number of sets of values of the unknown quantities that enter it; that is, it is indeterminate.

Every group of equations that contains more unknown quantities than there are equations, is indeterminate.

For, we may, by combination and elimination, reduce them to a single equation containing more than one unknown quantity, which we have already seen is indeterminate.

Dependent Equations.—If we have a greater number of equations than there are unknown quantities, they cannot all be satisfied, unless some of them are dependent on the others. If we combine them, and eliminate all the unknown quantities, the resulting equations, which will then contain only known quantities, will be *equations of condition*—that is, which must be satisfied in order that the given equations may admit of solution.

For example, if we have

$$x + y = a, \qquad x - y = c, \qquad \text{and} \quad xy = d;$$

we may combine the first two, and find,

$$x = \frac{a}{2} + \frac{c}{2} \quad \text{and} \quad y = \frac{a}{2} - \frac{c}{2};$$

by substituting these in the third, we find

$$\frac{a^2}{4} - \frac{c^2}{4} = d,$$

which expresses the relation between a, c, and d, that must exist, in order that the three equations may be simultaneous.

An Indeterminate Problem is one which admits of an infinite number of solutions. A problem will always be indeterminate when its enunciation involves more unknown quantities than there are given conditions; since, in that case, the statement of the problem will give rise to a less number of equations than there are unknown quantities.

EXAMPLES.

1. Let it be required to find two numbers such that 5 times the first diminished by 3 times the second shall be equal to 12.

If we denote the numbers by x and y, the conditions of the problem will give the equation

$$5x - 3y = 12,$$

which we have seen is indeterminate. Hence, the problem admits of an infinite number of solutions, or is indeterminate.

2. Find a quantity such that if it be multiplied by a and the product increased by b, the result will be equal to c times the quantity increased by d.

Let x denote the required quantity. Then, from the condition of the problem, we have

$$a + b = cx + d,$$

whence, $$x = \frac{d - b}{a - c}.$$

If we make the suppositions that $d = b$ and $a = c$, the value of x becomes $\frac{0}{0}$, which is a symbol of indetermination.

If we make these substitutions in the first equation, it becomes

$$ax + b = ax + b,$$

which is an identical equation (Art. **84**); that is, an equation that must be satisfied for all values of x. These suppositions render the conditions of the problem so dependent on each other, that any quantity whatever will satisfy them all.

Hence, the result $\frac{0}{0}$ indicates that a problem admits of an infinite number of solutions.

3. Find two quantities such that a times the first increased by b times the second shall be equal to c, and that d times the first increased by f times the second shall be equal to g.

If we denote the quantities by x and y, we shall have from the conditions of the problem,

$$ax + by = c \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$
$$dx + fy = g \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

whence, $$y = \frac{cd - ag}{bd - af} \quad \text{and} \quad x = \frac{bg - cf}{bd - af}$$

If now we make

$$cd = ag, \quad . \; . \; . \; (3) \qquad \text{and} \qquad af = bd, \quad . \; . \; . \; (4)$$

we shall find by multiplying these equations together, member by member,

$$cf = bg.$$

These suppositions reduce the values of both x and y to $\frac{0}{0}$.

From (3) we find, $d = \frac{ag}{c}$, and from (4) $f = \frac{b}{a} \times d = \frac{bg}{c}$, which substituted in (2), reduce it to

$$ax + by = c,$$

an equation identical with (1).

The suppositions made, render the conditions of the problem so dependent on each other, that there is, in reality, but one equation, which ought, therefore, to give an *indeterminate* result.

The result $\frac{0}{0}$, with the exception of the cases considered in Art. **80**, generally arises from some supposition on the quantities entering a problem, which makes the conditions of the problem so dependent on each other as to give rise to more unknown quantities than there are independent conditions. In these cases the result $\frac{0}{0}$ is a true answer to the problem; it indicates that the problem admits of an infinite number of solutions.

VIII. INTERPRETATION OF NEGATIVE RESULTS.

99. From the nature of the signs + and —, as explained in Art. **7**, we must give to a quantity preceded by the sign —, an interpretation diametrically opposite to that which it would receive were it preceded by the sign +. This will be better illustrated by a few

EXAMPLES.

1. To find a number, which added to b, will give a sum equal to a.

Let x denote the required number. Then from the conditions

$$x + b = a, \quad \text{whence,} \quad x = a - b.$$

This *formula* gives the algebraic value of x for all cases of the problem.

If we make $a = 47$ and $b = 29$, we have $x = +18$. In this case x is $+$, and we interpret it as requiring that 18 shall actually be *added* to 29 to produce 47.

If we make $a = 24$ and $b = 31$, we have $x = -7$. In this case x is $-$, and we interpret it as showing that 7 is to be *subtracted* from 31 to make the algebraic sum equal to 24.

2. At a certain epoch a father has lived a years, and his son b years. Find the number of years from the epoch when the father is *four* times as old as the son.

Let x denote the number of years required.

Then will $a + x$ be the age of the father,
and $b + x$ the age of the son, at the time required.

From the conditions of the problem, we have,

$$\frac{a + x}{4} = b + x; \qquad \therefore \qquad x = \frac{a - 4b}{3}.$$

If we suppose $a = 54$ and $b = 9$, we have $x = +6$. We agree to call time after the epoch $+$, and consequently interpret this result as showing that the required time is b years after the epoch—a conclusion which is readily verified.

If we make $a = 45$ and $b = 15$, we have $x = -5$. Since the time is $+$, *after* the epoch; it must be $-$, *before* the epoch. Hence, we interpret the negative result as showing that the father was four times as old as the son five years *before* the epoch. This conclusion is also easily verified.

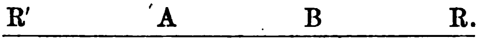
IX. DISCUSSION OF PROBLEMS.

100. The **Discussion** of a problem consists in making every possible supposition on the *arbitrary* quantities that enter it, and interpreting the results.

An **Arbitrary Quantity** is a quantity to which a value may be given at pleasure.

The discussion of the following problem indicates the general method of discussion, as well as illustrates the meaning of the *plus* and *minus* signs, as signs of *interpretation*, together with the ideas to be attached to the symbols, 0, ∞, and $\frac{0}{0}$.

101. Problem of the Couriers.—Two couriers, A and B, travel along the same line, in the same direction, R′ R, and at uniform rates; the courier A travels m miles per hour, and the courier B, n miles per hour. Now, supposing them to be separated by a distance a, at any epoch, say 12 o'clock, when are they together?



Let the position of the rearmost courier, A, be taken as the origin of distances, and suppose all distances estimated toward B to be positive.

Denote the number of hours from the epoch to the time they are together, by t, and the distance the courier B travels in the time t, by x; then will the distance the courier A travels, in the same time, be denoted by $a + x$.

Since the distance traveled is equal to the number of hours multiplied by the rate per hour, we have the equations,

$$mt = a + x, \qquad \text{and} \qquad nt = x;$$

whence, by solving

$$t = \frac{a}{m - n}, \quad x = \frac{na}{m - n}, \quad \text{and} \quad x + a = \frac{ma}{m - n}.$$

To discuss these values. The distance between the couriers at the epoch may be assumed at pleasure, but from the suppositions already made it cannot be negative. The rates of travel may also be assumed at pleasure. Hence, a, m, and n are *arbitrary*. The only hypotheses that can be made on a, are $a > 0$ and $a = 0$; the only hypotheses that can be made on m and n are $m > n$, $m < n$, and $m = n$. By combining these, we have the following six cases:

1°. $a > 0$ and $m > n$.
2°. $a > 0$ and $m < n$.
3°. $a > 0$ and $m = n$.
4°. $a = 0$ and $m > n$.
5°. $a = 0$ and $m < n$.
6°. $a = 0$ and $m = n$.

We shall make each hypothesis separately.

1°. $a > 0$ and $m > n$.

This hypothesis makes both terms of the fraction $\frac{a}{m - n}$ positive; hence, the value of t is positive.

We interpret this result as showing that the time required is *after* the epoch 12 o'clock.

This interpretation is in accordance with the supposition; for, if $m > n$, the courier A travels faster than the courier B; hence, after the epoch, the distance between them continually diminishes, and, consequently, at some time after the epoch they must be together.

The same hypothesis makes both x and $x + a$ positive, showing, as before, that the couriers are together somewhere to the right of the origin of distances.

2°. $a > 0$ and $m < n$.

This hypothesis makes the numerator of the fraction $\frac{a}{m - n}$ positive, and the denominator negative; hence, the value of t is *negative*.

We interpret this result as showing that the time when they are together is *before* the epoch.

This interpretation is also in accordance with the supposition; for, if $m < n$, the courier B travels faster than the courier A; hence, after the epoch, they continually separate. Before the epoch, they must have been together at some time, *previous* to which B was approaching A, and after which the two were continually separating.

This hypothesis makes both x and $x + a$ negative, showing, as before, that the couriers are together at a point R′, on the left of the origin of distances.

3°. $a > 0$ and $m = n$.

This hypothesis makes the numerator of the fraction $\frac{a}{m - n}$ finite, and the denominator 0; hence, the value of t is equal to a finite quantity divided by 0, or to ∞ (Art. **80**).

This result is interpreted as showing that the time from the epoch when the couriers are together is greater than any assignable time; that is, they are never together.

This interpretation is also in accordance with the hypothesis; for, if $m = n$, the couriers travel equally fast; and since they are separated by a distance a, at the epoch, they always have been and always will be separated by the same interval; that is, they can never be together.

The same supposition makes x and $x + a$, each equal to ∞, showing that the distance from the origin to the point where the couriers are together, is greater than any assignable distance.

4° and 5°. $a = 0$, $m > n$, and $a = 0$, $m < n$.

Either of these hypotheses makes the numerator of the fraction $\frac{a}{m - n}$ zero, and the denominator finite; hence, the value of t is equal to 0 divided by a finite quantity, or to 0 (Art. **80**).

This result is interpreted as showing that they are together at the epoch.

This interpretation is in accordance with the supposition; for, since $a = 0$, they are together at the epoch; and since m and n are unequal, they travel at different rates; consequently, they can never be together after the epoch, nor could they ever have been together before the epoch.

The same hypothesis makes both x and $x + a$ equal to 0, showing that the distance from the origin to the point where the couriers are together, is 0.

6°. $a = 0$, and $m = n$.

This hypothesis makes both terms of the fraction $\frac{a}{m - n}$ equal to 0; hence, the value of t is $\frac{0}{0}$, or, indeterminate (Art. **80**).

This result we interpret as showing that there is an *infinite number of times* when they are together.

This interpretation is also in accordance with the supposition made; for, a being 0, they are together at the epoch; and m being equal to n, they travel at the same rate; consequently, they will be together at all times, both before and after the epoch, 12 o'clock.

The same hypothesis renders the values of x and $x + a$ indeterminate, showing that there is an infinite number of distances, from the origin to the points where they are together.

102. The formula,

$$t = \frac{a}{m - n},$$

may be employed in many cases. As an example, let it be required to solve problem 33 of Article **91**. In this case, a is the number of minute spaces the minute hand must gain on the hour hand from 12 o'clock; hence, $a = 61$. The minute hand moves over 60 of these spaces in 1 hour, and the hour hand moves over 5 of them; hence, $m = 60$ and $n = 5$.

Substituting these values, we have,

$$t = \frac{61}{60 - 5}, \quad \text{or} \quad t = 1\tfrac{6}{55}h., \quad \text{or} \quad t = 1h.\ 6\tfrac{6}{11}m.$$

Monday 1-21-78.

X. OF INEQUALITIES.

103. **An Inequality** is an algebraic expression of two unequal quantities, connected by the sign of inequality. Thus, $a > b$ is an inequality, showing that a is greater than b.

Of two unequal quantities, that one is the greater which is nearer to $+\infty$, and that one is the less which is nearer to $-\infty$. Hence, of two negative quantities, that one is the greater, algebraically, which has the fewer units.

The part on the left of the sign is called the *first member*, and the part on the right, the *second member* of the inequality.

Two inequalities subsist in the *same* sense when the greater quantity is in the first member of both, or in the second member of both; they subsist in a *contrary* sense when the greater quantity is in the first member of one and in the second member of the other. Thus, the inequalities

$$25 > 30 \quad \text{and} \quad 18 > 10,$$

subsist in the same sense; and the inequalities

$$15 > 13 \quad \text{and} \quad 12 < 14$$

subsist in a contrary sense.

The following principles enable us to transform inequalities:

1°. *If we add the same quantity to, or subtract the same quantity from, both members of an inequality, the resulting inequality will subsist in the same sense.*

Thus, if we add 5 to, or subtract 5 from both members of the inequality

$$4 > 2,$$

we have,

$$9 > 7 \quad \text{and} \quad -1 > -3.$$

This principle enables us to transpose a term from one member of an inequality to the other by simply changing its sign. Thus, from the inequality

$$3x - b > 2x + a,$$

we find, by transposition,

$$x > a + b.$$

2°. *If two members of an inequality be multiplied or divided by a positive quantity, the resulting inequality will subsist in the same sense.*

Thus, if we multiply or divide both members of the inequality

$$12 > 8$$

by $+4$, we have

$$48 > 32 \quad \text{and} \quad 3 > 2.$$

This principle enables us to clear an inequality of fractions. Thus, if we multiply both members of the inequality

$$\frac{3x}{4} - \frac{2b}{3} > \frac{x}{6} - 7$$

by 12, and reduce, we have

$$9x - 8b > 2x - 84.$$

The following precautions are to be observed in treating inequalities:

3°. *If both members of an inequality are positive, they may be raised to like powers without changing the sense of the inequality.*

Thus, if both members of the inequality

$$12 > 5$$

be squared or cubed, we have

$$144 > 25 \quad \text{and} \quad 1728 > 125.$$

4°. *If the two members of an inequality be multiplied or divided by a negative quantity, the sense of the inequality must be reversed.*

Thus, if we multiply or divide both members of the inequality

$$3 < 7$$

by -4, we have

$$-12 > -28.$$

5°. *If both members of an inequality are negative, they may be raised to any odd power, and the resulting inequality will subsist in the same sense; but if both members be raised to an even power, the sense of the resulting inequality will be reversed.*

Thus, if we square and cube both members of the inequality

$$-3 > -5,$$

we have

$$9 < 25 \qquad \text{and} \qquad -27 > -125.$$

By the aid of these principles we may solve an inequality; that is, we can find an inequality in which the unknown quantity shall form one member.

EXAMPLES.

1. $5x - 6 > 19.$ *Ans.* $x > 5.$

2. $3x + \frac{14}{2}x - 30 > 10.$ *Ans.* $x > 4.$

3. $\frac{x}{6} - \frac{1}{3}x + \frac{x}{2} + \frac{13}{2} > \frac{17}{2}.$ *Ans.* $x > 6.$

4. $\frac{ax}{5} + bx - ab > \frac{a^2}{5}.$ *Ans.* $x > a.$

5. $\frac{bx}{7} - ax + ab < \frac{b^2}{7}.$ *Ans.* $x < b.$

Tuesday 1-22-78

CHAPTER V.

SQUARE ROOT AND RADICALS OF THE SECOND DEGREE.

I. Square Root of Numbers.

104. The Square of a Number is the product that arises from taking that number twice as a factor. Thus, 144 is the square of 12, because, 12 × 12 = 144.

The Square Root of a Number is one of its *two* equal factors. Thus, 12 is the square root of 144, because, if 144 be resolved into two equal factors, one of those factors is 12.

When a number can be resolved into two equal factors, which are entire, it is said to be a *perfect square*, and its square root can be *exactly* found. When a number cannot be resolved into two equal entire factors, it is said to be an *imperfect square*, and its square root can only be found *approximately*. Thus, 144 is a perfect square, and 12 is its exact square root; but 40 is an imperfect square, and 6.324 is its approximate square root.

To deduce a rule for finding the square root of a whole number, let us write down the first ten whole numbers and their corresponding squares, as in the following

TABLE.

Numbers,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10;
Squares,	1,	4,	9,	16,	25,	36,	49,	64,	81,	100.

When a number is less than 100, look for it in the lower line of the table; if it is found there, the corresponding number in the upper line is its square root; if it is not found there, it will lie between two consecutive numbers of that line, and its square root will lie between the corresponding numbers in the upper line.

When a number is greater than 100, its square root will be greater than 10; that is, it will contain more than one place of figures. In this case, denote the number by N, and let a denote the *tens* and b the *units* of its square root; we shall have,

$$N = (a + b)^2 = a^2 + 2ab + b^2.$$

Hence, every number is made up of *the square of the tens* of its root, plus *twice the product of the tens by the units,* plus *the square of the units.*

$$\begin{array}{r|l}
 & 6\dot{0}\,8\dot{4}\ \underline{|78} \\
 & \underline{49} \\
7 \times 2\,..\,148 & 1184 \\
 & \underline{1184} \\
 & 0
\end{array}$$

This law enables us to find the square root of any number.

1. Let us take the number 6084. Since the square of the tens cannot contain any significant figure of a less denomination than hundreds, we point off a period of two figures on the right, and look for the square of the tens in what remains. The greatest perfect square in 60 is 49, and its square root is 7; hence, there are 7 tens in the required root. Subtracting 49 from 60, and bringing down the period 84, we have for a remainder 1184; this is made up of *twice the product of the tens by the units,* plus *the square of the units;* hence, if we divide it by *twice the tens*—that is, by 140—or, what is the same thing, if we divide 118 by 14, the quotient 8 will be the number of units, or some greater number. As a test, we annex 8 to the trial divisor 14, and also to the root 7, and then multiply it into the augmented divisor 148; this multiplication gives *twice the product of the tens by* 8, plus *the square of* 8; and since this is found equal to 1184, we infer that 8 is the required number of units. Hence, the entire root is 78. Had the last multiplication

given a number greater than 1184, we *should have diminished the figure to be tested until the product was equal to or less than this number.*

72. Let us next extract the square root of a number containing more than four figures.

$$\begin{array}{r|l|l}
 & 5\dot{6}\;8\dot{2}\;1\dot{4}\;4\dot{4} & \underline{7538} \\
 & 49 & \\
1\dot{4}\,5 & 7\dot{8}\,2 & \\
 & \underline{72\,5} & \\
15\dot{0}\,3 & 57\dot{1}\,4 & \\
 & \underline{450\,9} & \\
150\dot{6}\,8 & 1205\dot{4}\,4 & \\
 & \underline{12054\,4} &
\end{array}$$

Let 56821444 be the number.

If we consider the root as made up of a certain number of tens and a certain number of units, the number will, as before, be equal to the square of the tens plus twice the product of the tens by the units plus the square of the units.

If then, as before, we point off a period of two figures, at the right, the square of the tens of the required root will be found in the number 568214, at the left; and the square root of the greatest perfect square in this number will express the tens of the root.

But since this number, 568214, contains more than two figures, its root contains more than one; that is, the square root of the given number contains *hundreds*, and the square of the hundreds will be found in the figures 5682; hence, if we point off a second period of two figures, the square root of the greatest perfect square in what remains will be the hundreds of the required root. But since 5682 contains more than two figures, its root will contain more than one; that is, the root of the given number contains thousands, and the square of the thousands will be found in 56; hence, if we point off a third period of two figures, the square root of the greatest perfect square in what remains will be the thousands of the required root. Hence, placing a point over 56, we proceed thus:

The greatest perfect square in the first period is 49, and its square root is 7; hence, 7 is the number of thousands of the required root. Placing 7 on the right of the given number, and subtracting its square, 49, from the left-hand period, we find 7 for a remainder, to which we annex the next period, 82. Separating the last figure on the right from the others by a point, and dividing the number on the left by twice 7, or 14, we have 5 for a quotient, which we place at the right of the figure already found, and also annex it to 14. Multiplying 145 by 5, and subtracting the product from 782, we find the remainder 57. Hence, 75 is the number of hundreds of the required square root. Then bring down the next period, annex it to the second remainder, giving 5714, and divide 571 by twice 75, or 150. The quotient 3 annexed to 75 gives 753 for the number of tens in the root sought; find, as before, the number of units, which in this case is 8. Hence, the required square root is 7538.

A similar course of reasoning may be applied to any other number. Hence, for the extraction of the square root of whole numbers, we have the following

RULE.

I. Separate the given number into periods of two figures each, beginning at the right hand: the period on the left will often contain but one figure:

II. Find the greatest perfect square in the first period on the left, and place its root on the right, after the manner of a quotient in division. Subtract the square of this root from the first period, and to the remainder bring down the second period for a dividend:

III. Double the root already found and place it on the left for a divisor. See how many times the divisor is contained in the dividend, exclusive of the right-hand figure, and place the quotient in the root and also at the right of the divisor:

IV. Multiply the divisor, thus augmented, by the last figure of the root found, and subtract the pro-

duct from the dividend, and to the remainder bring down the next period for a new dividend:

V. Double the whole root already found, for a new divisor, and continue as before, until all the periods are brought down and operated upon.

If the final remainder is 0, the given number is a perfect square; if it is not 0, the root found is the *entire part* of the required root.

It is to be observed that the root contains as many places of figures as the given number contains periods.

EXAMPLES.

1. Find the square root of 7225. *Ans.* 85.
2. Find the square root of 17689. *Ans.* 133.
3. Find the square root of 994009. *Ans.* 997.
4. Find the square root of 85678973. *Ans.* 9256+.
5. Find the square root of 67812675. *Ans.* 8234+.
6. Find the square root of 2792401. *Ans.* 1671+.
7. Find the square root of 37496042. *Ans.* 6123+.
8. Find the square root of 3661097049. *Ans.* 60507.
9. Find the square root of 918741672704. *Ans.* 958510+.
10. Find the square root of 74684164. *Ans.* 8642.

105. Imperfect Squares.—The square root of a whole number that is an imperfect square, is incommensurable with 1; that is, its value cannot be expressed in exact parts of 1. For, let c be a whole number that is an imperfect square, and suppose that its square root is equal to $\frac{a}{b}$, in which a and b are prime whole numbers. We shall have,

$$\sqrt{c}=\frac{a}{b}; \qquad \therefore \qquad c=\frac{a^2}{b^2};$$

but, by hypothesis a and b, have no common factor; hence, a^2 and b^2 have no common factor; that is, we have a whole number equal to an irreducible fraction, which is absurd.

G. W. McDonnell

Hence, it is impossible that the square root of an imperfect square should be expressed in exact parts of 1.

In like manner, it may be shown that *any root of an imperfect power cannot be expressed in exact parts of* 1.

106. Fractions.—From the principle of multiplication we see that a fraction can be squared by squaring its numerator and denominator separately. Reversing the operation, we find the following rule for extracting the square root of a fraction:

RULE.

Extract the square root of the numerator for a new numerator, and the square root of the denominator for a new denominator.

EXAMPLES

Extract the square root of the following fractions:

1. $\frac{4096}{582169}$. *Ans.* $\frac{64}{763}$.
2. $\frac{273529}{956484}$. *Ans.* $\frac{523}{978}$.
3. $54\frac{19}{25}$ or $\frac{1369}{25}$. *Ans.* $7\frac{2}{5}$.

Approximation.—If the numerator and denominator are not both perfect squares, the exact root of the fraction cannot be exactly found. We can, however, find the root to within less than the fractional unit. Thus, to extract the square root of $\frac{a}{b}$ to within less than $\frac{1}{b}$, multiply both terms of the fraction by b, and we have $\frac{ab}{b^2}$.

Let r^2 be the greatest perfect square in ab, then will ab be contained between r^2 and $(r+1)^2$, and $\frac{ab}{b^2}$ will be contained between

$\frac{37}{5}$

$$\frac{r^2}{b^2} \quad \text{and} \quad \frac{(r+1)^2}{b^2},$$

and the square root of $\frac{ab}{b^2}$, or its equal $\frac{a}{b}$, will be contained between

$$\frac{r}{b} \quad \text{and} \quad \frac{r+1}{b}.$$

But the difference between $\frac{r}{b}$ and $\frac{r+1}{b}$, is $\frac{1}{b}$; hence, either will be the square root of $\frac{a}{b}$ to within less than $\frac{1}{b}$. We have, then, the following

RULE.

Multiply the numerator by the denominator, and extract the square root of the product to within less than 1*; divide the result by the denominator, and the quotient will be the approximate root.*

For example, to extract the square root of $\frac{3}{5}$, we multiply 3 by 5, which gives 15; the perfect square nearest 15, is 16, and its square root is 4; hence, $\frac{4}{5}$ is the square root of $\frac{3}{5}$ to within less than $\frac{1}{5}$.

107. Approximation for Integers.—To determine the square root of a whole number which is an imperfect square, to within less than a given fractional unit, as $\frac{1}{b}$

Reduce the number to a fractional form, having the given fractional unit, and then apply the preceding rule: or what is an equivalent,

Multiply the given number by the square of the denominator of the fraction which determines the degree of approximation; then extract the square root of the product to the *arest unit, and divide this root by the denominator of the* *tion.*

EXAMPLES.

1. Let it be required to extract the square root of 59, to within less than $\frac{1}{12}$

First, $(12)^2 = 144$; and $144 \times 59 = 8496$.

Now, the square root of 8496, to the nearest unit, is 92: hence,

$$\frac{92}{12} = 7\tfrac{8}{12}, \text{ which is true to within less than } \frac{1}{12}.$$

2. Find the $\sqrt{11}$ to within less than $\frac{1}{15}$. *Ans.* $3\frac{4}{15}$.

3. Find the $\sqrt{223}$ to within less than $\frac{1}{40}$. *Ans.* $14\frac{37}{40}$.

To find the square root of a whole number to within $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, etc., it is only necessary, according to the preceding rule, to multiply the number by $(10)^2$, $(100)^2$, $(1000)^2$; or, what is the same thing,

Annex to the number, two, four, six, etc., ciphers: then extract the root of the result to the nearest unit, and divide this root by 10, 100, 1000, etc., *which is effected by pointing off one, two, three, etc., decimal places.*

EXAMPLES.

1. To find the square root of 7 to within less than $\frac{1}{100}$.

Having multiplied by $(100)^2$; that is, having annexed four ciphers to the right hand of 7, it becomes 70000, whose root to the nearest unit, is 264; this divided by 100 gives 2.64 for the answer, and this is true to within less than $\frac{1}{100}$.

```
      7 00 00 | 2.64
      4       |
     ----
 46 | 300
    | 276
    -------
524 |  2400
    |  2096
    -------
        304 Rem.
```

2. Find the $\sqrt{29}$ to within less than $\frac{1}{100}$. *Ans.* 5.38.

3. Find the $\sqrt{227}$ to within less than $\frac{1}{10000}$. *Ans.* 15.0665.

The number of ciphers to be annexed is always equal to double the number of decimal places required to be found in the root.

108. Entire and Decimal.—The manner of extracting the square root of a number containing an entire part and decimals, follows immediately from the preceding.

Let us take for example the number 3.425. This is equivalent to $\frac{3425}{1000}$. Now, the denominator is not a perfect square, but may be made such without altering the value of the fraction, by multiplying both terms by 10; this gives

$$\frac{34250}{10000} \quad \text{or} \quad \frac{34250}{(100)^2}.$$

Then, extracting the square root of 34250 to the nearest unit, we find 185; hence, $\frac{185}{100}$ or 1.85 is the required root to within less than $\frac{1}{100}$.

If greater exactness is required, annex as many ciphers as will make the number of periods of decimals equal to the number of decimal places required. Hence, to extract the square root of a mixed decimal:

Annex ciphers until the whole number of decimal places is equal to double the number required in the root. Then, extract the root to the nearest unit, and point off the required number of decimal places.

EXAMPLES.

1. Find the $\sqrt{3271.4707}$ to within less than .01. *Ans.* 57.19.

2. Find the $\sqrt{31.027}$ to within less than .01.

Ans. 5.57.

3. Find the $\sqrt{0.01001}$ to within less than .00001.

1-25-78

Ans. 0.10004.

Vulgar Fractions.—To find the square root of a vulgar fraction in terms of decimals:

Change the vulgar fraction to a decimal and continue the division until the number of decimal places is double the number required in the root. Then, extract the root of the decimal by the last rule.

EXAMPLES.

1. Extract the square root of $\frac{11}{14}$ to within less than .001.

This number, reduced to decimals, is 0.785714 to within less than 0.000001. The root of 0.785714, to the nearest unit, is 886; hence, 0.886 is the root of $\frac{11}{14}$ to within less than .001.

2. Find the $\sqrt{2\frac{13}{15}}$ to within less than 0.0001. *Ans.* 1.6931.

II. SQUARE ROOT OF ALGEBRAIC QUANTITIES.

109. Monomials.—To deduce a rule for extracting the square root of a monomial, let us examine the law of formation of its square. Let us take, as an example, the monomial $5a^2b^3c$. By the rule for multiplication of monomials, we have,

$$(5a^2b^3c)^2 = 5a^2b^3c \times 5a^2b^3c = 25a^4b^6c^2;$$

that is, to square a monomial, we *square its coefficient, and double the exponent of each letter.*

Reversing this operation, we have for extracting the square root of a monomial the following

RULE.

Extract the square root of the coefficient for a new coefficient, and write after this, each letter, with an exponent equal to its original exponent divided by two.

A monomial is a perfect square when its coefficient is a perfect square and the exponents of all its letters are even numbers.

EXAMPLES.

Extract the square root of the following:

1. $16a^2b^2c^4$. *Ans.* $4abc^2$.
2. $81a^4b^4$. *Ans.* $9a^2b^2$.
3. $625a^8b^{10}$. *Ans.* $25a^4b^5$.
4. $9a^{2m-2}c^2$. *Ans.* $3a^{m-1}c$.
5. $16a^{-2}b^{-6}$. *Ans.* $4a^{-1}b^{-3}$.
6. $\frac{81}{4}a^{4n-2}d^6c^{2p}$. *Ans.* $\frac{9}{2}a^{2n-1}d^3c^p$.
7. $36x^{-\frac{2m}{n}}y^{-\frac{2n}{m}}$. *Ans.* $6x^{-\frac{m}{n}}y^{-\frac{n}{m}}$.
8. $289x^{-2}y^{-4}z^2$. *Ans.* $17x^{-1}y^{-2}z$.

√ W 9-78

109. Polynomials.—To deduce a rule for extracting the square root of a polynomial, let us first consider the law of formation of the square of a polynomial, taking for this purpose the polynomial,

$$2x^2 + 3x + 1, \text{ or, } (2x^2 + 3x) + 1.$$

Considering the bracketed binomial as one quantity, we have (Art. **33**, 1°),

$$[(2x^2 + 3x) + 1]^2 = (2x^2 + 3x)^2 + 2(2x^2 + 3x) + 1;$$

developing and reducing, we have,

$$(2x^2 + 3x + 1)^2 = 4x^4 + 12x^3 + 13x^2 + 6x + 1.$$

Reversing the operation, we find the square root by a process entirely analogous to that used for extracting the square root of whole numbers. Thus,

$$
\begin{array}{r|l|l}
 & 4x^4+12x^3+13x^2+6x+1 & \underline{2x^2+3x+1} \\
 & \underline{4x^4 \qquad\qquad\qquad\qquad\qquad} & \\
4x^2+3x & +12x^3+13x^2+6x+1 & \\
 & \underline{\;\;12x^3+\;\;9x^2 \qquad\qquad\quad} & \\
 & 4x^2+6x+1 \;\big|\; 4x^2+6x+1 & \\
 & \qquad\qquad\qquad\;\;\big|\;\underline{4x^2+6x+1} & \\
 & \qquad\qquad\qquad\qquad\quad 0 &
\end{array}
$$

Here we take the square root of $4x^4$, which is $2x^2$, for the first term of the root, and then subtract $4x^4$ from the given polynomial, for a remainder.

We then double $2x^2$ for a partial divisor, and find how many times it is contained in $12x^3$; the quotient, $3x$, we add to the partial divisor, and also to the root, and then multiply the divisor thus augmented by this last term, subtracting the product from the first remainder, for a second remainder. Again doubling the root already found, and proceeding as before, we find 1 for the third term of the root, and 0 for a final remainder.

Since all other cases may be treated in a similar manner, we have the following

RULE.

I. Arrange the polynomial with respect to one of its letters, and extract the square root of the first term for the first term of the root. Subtract the square of this term from the polynomial, for the first remainder:

II. Double the root already found, and place it on the left of the first remainder for a divisor; divide the first term of the first remainder by this divisor, for the second term of the root; add the quotient to the root found, and also to the divisor; multiply the *divisor, thus augmented, by the last term of the root found, and subtract the product from the first remainder for a second remainder:*

III. Double the root already found for a second divisor. Divide the first term of the second remainder by the first term of the second divisor for the third term of the root; add this term to the root and to the second divisor, and proceed as before, continuing the operation as far as desirable.

If a final remainder is found, equal to 0, the root is exact. If no such remainder is found, the polynomial is not a perfect square.

EXAMPLES.

Find the square roots of the following polynomials:

1. $25a^4 - 30a^3b + 49a^2b^2 - 24ab^3 + 16b^4$. *Ans.* $5a^2 - 3ab + 4b^2$.

2. $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$. *Ans.* $a^2 + 2ax + x^2$.

3. $a^4 - 2a^3x + 3a^2x^2 - 2ax^3 + x^4$. *Ans.* $a^2 - ax + x^2$.

4. $4x^6 + 12x^5 + 5x^4 - 2x^3 + 7x^2 - 2x + 1$. *Ans.* $2x^3 + 3x^2 - x + 1$.

5. $9a^4 - 12a^3b + 28a^2b^2 - 16ab^3 + 16b^4$. *Ans.* $3a^2 - 2ab + 4b^2$.

6. $25a^4b^2 - 40a^3b^2c + 76a^2b^2c^2 - 48ab^2c^3 + 36b^2c^4 - 30a^4bc + 24a^3bc^2 - 36a^2bc^3 + 9a^4c^2$. *Ans.* $5a^2b - 4abc + 6bc^2 - 3a^2c$.

7. $x^4 + 2ax^3 + 3a^2x^2 + 2a^3x + a^4$. *Ans.* $x^2 + ax + a^2$.

8. $25x^4y^2 - 30x^3y^3 + 29x^2y^4 - 12xy^5 + 4y^6$. *Ans.* $5x^2y - 3xy^2 + 2y^3$.

9. $x^6 - 4x^5 + 4x^4 + 12x^3 - 24x^2 + 36$. *Ans.* $x^3 - 2x^2 + 6$.

10. $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16}$. *Ans.* $x^2 - x + \frac{1}{4}$.

$x^2 + \frac{2ax}{3} - bx + \frac{a^2}{9} + \frac{b^2}{4} - \frac{ab}{3}$. *Ans.* $x + \frac{a}{3} - \frac{b}{2}$.

12. $4x^4 + 6x^3 + \frac{89}{4}x^2 + 15x + 25.$ *Ans.* $2x^2 + \frac{3}{2}x + 5.$

13. $a^{2m} - 4a^m b^{3n} + 4b^{6n}.$ *Ans.* $a^m - 2b^{3n}.$

14. $\frac{a^4}{4} + a^3x + \frac{4a^2x^2}{3} + \frac{2ax^3}{3} + \frac{x^4}{9}.$ *Ans.* $\frac{a^2}{2} + ax + \frac{x^2}{3}.$

15. $x^2 + \frac{1}{x^2} + 2x - \frac{2}{x} - 1.$ *Ans.* $x + 1 - \frac{1}{x}.$

Principles.—1°. A binomial can never be a perfect square. For, its root cannot be a monomial, since the square of a monomial is a monomial; nor can its root be a polynomial, since the square of the simplest polynomial, viz., a binomial, contains at least three terms. Thus, an expression of the form

$$a^2 \pm b^2$$

can never be a perfect square.

2°. A trinomial may be a perfect square. If so, when arranged, its extreme terms are squares, and the middle term is double the product of the square roots of the other two. Therefore, to obtain the square root of a trinomial, when it is a perfect square;

Extract the square roots of the extreme terms, and give these roots the same or contrary signs, according as the middle term is positive or negative.

Thus, $9a^6 - 48a^4b^2 + 64a^2b^4$, is a perfect square, and its square root is,

$$3a^3 - 8ab^2.$$

III. RADICALS OF THE SECOND DEGREE.

110. **The Radical Sign** $\sqrt{}$, indicates a root to be extracted (Art. **6**, 6°). When the indicated root can be found exactly, the quantity is said to be a *rational quantity* under a radical form: thus, $\sqrt{16}$ is a rational quantity under a radical form.

A Radical Quantity is an indicated root of an imperfect power of the degree indicated: thus, $\sqrt{7}$ and $\sqrt{3ab}$, are radical quantities: they are also called, *Irrational Quantities*, or *Surds*.

Degrees of Radicals.—Radicals are also divided into *degrees*, according to the index of the root to be extracted.

When the square root is indicated, the radical is of the *second* degree; when the cube root is indicated, the radical is of the *third* degree; and so on. Thus, $\sqrt{7}$, $\sqrt{3}$, and $\sqrt{a}$ are radicals of the *second* degree; $\sqrt[3]{a}$ and $\sqrt[3]{16}$ are radicals of the *third* degree.

The *index of the root* is called the *index of the radical*.

Similar Radicals.—Radicals are *similar* when they are of the same degree, and when the quantity under the radical sign is the same in both. Thus, $a\sqrt{b}$ and $c\sqrt{b}$ are similar. The quantity written before the radical sign is called the *coefficient of the radical*. Thus, in the radicals just given, a and c are coefficients, each showing how many times $\sqrt{b}$ is taken.

111. Sign of the Root.—Since the product of two negative quantities is +, as well as the product of two positive ones, it follows that the square of $-a$, as well as the square of $+a$, is $+a^2$; hence, the square root of a^2 is both + and −; that is, a^2 has *two* square roots, $+a$ and $-a$. This result may be written,

$$\sqrt{a^2} = \pm a.$$

In all cases a *positive* quantity has two square roots, numerically equal, but with contrary signs. From what has just been shown, the square of a positive quantity is +, and the square of a negative quantity is also +; hence, it is impossible that the square of any real quantity be negative. ence, the indicated square root of a negative quantity is l *imaginary*. Thus, $\sqrt{-2}$, $\sqrt{-7}$, and $\sqrt{-16}$ are imaginary quantities, or imaginary expressions. Any indicated root of an even degree, of a negative quantity, is imaginary.

It will be shown hereafter that imaginary expressions admit of interpretation, and that they are subject to most of the operations of Algebra.

A root of an odd degree, of a negative quantity, is *real;* thus,

$$\sqrt[3]{-27} = -3.$$

112. Transformation.—The *transformation* of a radical is the operation of changing its form without changing its value. The principal transformations of radicals of the second degree depend on the following principles·

1°. Let a and b denote any two quantities, and p the product of their square roots; then,

$$\sqrt{a} \times \sqrt{b} = p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Squaring both members of equation (1), we have (Art. **86**, Ax. 5),

$$(\sqrt{a})^2 \times (\sqrt{b})^2 = p^2,$$

or,

$$ab = p^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Extracting the square root of both members of equation (2), we have (Art. **86**, Ax. 6),

$$\sqrt{ab} = p;$$

but things that are equal to the same thing are equal to each other; therefore,

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}; \quad \text{hence,}$$

The product of the square roots of two quantities is equal to the square root of the product of those quantities; and the reverse.

2°. Denote the quotient of $\sqrt{a}$ by $\sqrt{b}$, by q; whence,

$$\frac{\sqrt{a}}{\sqrt{b}} = q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Squaring both members of equation (1), we find,

$$\frac{(\sqrt{a})^2}{(\sqrt{b})^2} = q^2,$$

or,

$$\frac{a}{b} = q^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Extracting the square root of both members of equation (2), we have

$$\sqrt{\frac{a}{b}} = q.$$

But things that are equal to the same thing are equal to each other; therefore,

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}; \quad \text{hence,}$$

The quotient of the square roots of two quantities is equal to the square root of the quotient of the same quantities; and the reverse.

113. Simplest Form.—A radical of the second degree is in its simplest form when there is no factor under the radical sign that is a perfect square.

To reduce the radical, $\sqrt{98ab^4}$, to its simplest form, we have,

$$\sqrt{98ab^4}, = \sqrt{49b^4 \times 2a}.$$

But from the first principle we have,

$$\sqrt{49b^4 \times 2a} = \sqrt{49b^4} \times \sqrt{2a} = 7b\sqrt{2a}.$$

Since all similar cases may be treated in the same manner, we have, for reducing a radical of the second degree to its simplest form, the following

RULE.

I. Resolve the quantity under the radical sign into two factors, one of which is the greatest perfect square that enters it, as a factor:

II. Write the square root of the perfect square before the radical sign, under which place the other factor.

EXAMPLES.

Reduce the following expressions to their simplest forms:

1. $\sqrt{75a^3bc}$. *Ans.* $5a\sqrt{3abc}$.

2. $\sqrt{128b^5a^6d^2}$. *Ans.* $8b^2a^3d\sqrt{2b}$.

3. $\sqrt{32a^9b^8c}$. *Ans.* $4a^4b^4\sqrt{2ac}$.

4. $\sqrt{256a^2b^4c^8}$. *Ans.* $16ab^2c^4$.

5. $\sqrt{1024a^9b^7c^5}$. *Ans.* $32a^4b^3c^2\sqrt{abc}$.

6. $\sqrt{728a^7b^5c^6d}$. *Ans.* $2a^3b^2c^3\sqrt{182abd}$.

Radical polynomials can be simplified in like manner when they can be factored.

7. $\sqrt{a^3b + 4a^2b^2 + 4ab^3}$. *Ans.* $(a + 2b)\sqrt{ab}$.

8. $\sqrt{2(ax - bx^2)^3}$. *Ans.* $(ax - bx^2)\sqrt{2(ax - bx^2)}$.

9. $\sqrt{x^3 - 12x - 16}$. *Ans.* $(x + 2)\sqrt{x - 4}$.

114. Reverse Process.—By reversing the above process, we may introduce a coefficient, or a factor of a coefficient, under the radical sign. Thus,

$$a\sqrt{b} = \sqrt{a^2} \times \sqrt{6} = \sqrt{a^2b}$$

And since the same operation may be performed in all similar cases, we have the following

RULE.

Square the coefficient or factor, and write it under the radical sign as a factor.

EXAMPLES.

Place the coefficients in the following examples under the radical sign:

1. $6\sqrt{13}$ *Ans.* $\sqrt{468}$.

2. $12\sqrt{7}$. *Ans.* $\sqrt{1008}$.

3. $a\sqrt{b-c}$. *Ans.* $\sqrt{a^2b-a^2c}$.

The principal use of this transformation is to find an approximate value of a radical to within a given unit. Thus to find the value of $6\sqrt{13}$, we may extract the square roo of 13 to within 1, which gives 4 as the nearest value; the multiplying by 6, we find 24, in which the error committe by taking 4 as the square root of 13 is multiplied by 6. I however, we introduce 6 under the radical sign, givin $\sqrt{468}$, and then extract the root to within 1, we have,

$$6\sqrt{13} = 21, \text{ to within less than } 1.$$

In like manner,

$$12\sqrt{7} = 31, \text{ to within less than } 1.$$

115. Addition and Subtraction.—If two radicals the second degree, when reduced to their simplest form (A **113**), are not similar (Art. **110**), they must be connec with each other by their proper signs. But if they are si lar, when so reduced, they may be added or subtracted the following

RULE.

Add or subtract the coefficients, and to the s or difference annex the common radical part.

EXAMPLES.

Add the following radicals:

1. $3a\sqrt{b}$, $5c\sqrt{b}$, and $2c\sqrt{b}$. *Ans.* $(3a+7c)$

2. $7\sqrt{2a}$, $3\sqrt{2a}$, and $\sqrt{8a}$. *Ans.* $12\sqrt{2a}$.

3. $\sqrt{48ab^2}$ and $b\sqrt{75a}$. *Ans.* $9b\sqrt{3a}$.

4. From $7\sqrt{2a}$ subtract $2\sqrt{2a}$. *Ans.* $5\sqrt{2a}$.

5. From $\sqrt{48ab^2}$ subtract $b\sqrt{75a}$. *Ans.* $-b\sqrt{3a}$.

6. From $\sqrt{98a^5b^3}$ subtract $\sqrt{18a^3b^4}$.

Ans. $(7a - 3b)\,ab\sqrt{2a}$.

116. Multiplication of Radicals.—Let $a\sqrt{b}$ and $c\sqrt{d}$ be any two radicals of the second degree; their product will be denoted thus,

$$a\sqrt{b} \times c\sqrt{d},$$

which, since the order of the factors may be changed without altering the value of the product, may be written,

$$a \times c \times \sqrt{b} \times \sqrt{d}.$$

The product of the last factors from the 1st principle of Art. 112. is equal to $\sqrt{bd}$; we have, therefore,

$$a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}.$$

Hence, to multiply one radical of the second degree by another, we have the following

RULE.

Multiply the coefficients together for a new coefficient; and the quantities under the radical sign for a new quantity under the radical sign.

EXAMPLES.

1. $3\sqrt{5ab} \times 4\sqrt{20a} = 12\sqrt{100a^2b} = 120a\sqrt{b}$.

2. $2a\sqrt{bc} \times 3a\sqrt{bc} = 6a^2\sqrt{b^2c^2} = 6a^2bc$.

3. $2a\sqrt{a^2 + b^2} \times -3a\sqrt{a^2 + b^2} = -6a^2(a^2 + b^2)$.

Perform the following indicated multiplications:

4. $(\sqrt{a} - 3\sqrt{b}) \times (\sqrt{a} - 2\sqrt{b})$. *Ans.* $a - 5\sqrt{ab} + 6b$.

5. $(5\sqrt{8} + 2\sqrt{5}) \times (3\sqrt{5} - 2\sqrt{2})$.
Ans. $-10 + 26\sqrt{10}$.

6. $2a\sqrt{bc} \times 3b\sqrt{abc} \times a\sqrt{2a}$. *Ans.* $6a^3b^2c\sqrt{2}$.

117. Division of Radicals.— Let $a\sqrt{b}$ and $c\sqrt{d}$ represent any two radicals of the second degree, and let it be required to find the quotient of the first by the second. This quotient may be indicated thus,

$$\frac{a\sqrt{b}}{c\sqrt{d}}, \text{ which is equal to } \frac{a}{c} \times \frac{\sqrt{b}}{\sqrt{d}};$$

but from the 2d principle of Art. **112**,

$$\frac{\sqrt{b}}{\sqrt{d}} = \sqrt{\frac{b}{d}}; \quad \text{consequently,} \quad \frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c}\sqrt{\frac{b}{d}}.$$

Hence, to divide one radical of the second degree by another, we have the following

RULE.

Divide the coefficient of the dividend by the coefficient of the divisor for a new coefficient; and the quantity under the radical sign in the dividend by the quantity under the radical sign in the divisor for a new quantity under the radical sign.

EXAMPLES.

1. Divide $5a\sqrt{b}$ by $2b\sqrt{c}$. *Ans.* $\frac{5a}{2b}\sqrt{\frac{b}{c}}$.

2. Divide $12ac\sqrt{6bc}$ by $4c\sqrt{2b}$. *Ans.* $3a\sqrt{3c}$.

3. Divide $25a^2b\sqrt{m^3n}$ by $5ab^2\sqrt{mn^2}$. *Ans.* $\frac{5am}{b}\sqrt{\frac{1}{n}}$

4. Divide $\sqrt{3a^2c + 6abc + 3b^2c}$ by $\sqrt{5a^2c - 10ac^2 + 5c^3}$.
Ans. $\frac{a+b}{a-c}\sqrt{\frac{3}{5}}$.

118. Rational Terms.—For the purposes of numerical computation, it is often desirable that the denominator of a radical fraction should be made rational.

When the denominator is a simple radical, we may multiply both terms of the fraction by the denominator. Thus,

$$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

Here the latter expression is more easily computed than the former.

When the denominator is a binomial radical, the denominator may be made rational by multiplying both terms by the denominator with the sign of its second term changed. Thus,

$$\frac{a}{\sqrt{p} + \sqrt{q}} = \frac{a(\sqrt{p} - \sqrt{q})}{(\sqrt{p} + \sqrt{q})(\sqrt{p} - \sqrt{q})} = \frac{a\sqrt{p} - a\sqrt{q}}{p - q};$$

also,

$$\frac{a}{\sqrt{p} - \sqrt{q}} = \frac{a(\sqrt{p} + \sqrt{q})}{(\sqrt{p} - \sqrt{q})(\sqrt{p} + \sqrt{q})} = \frac{a\sqrt{p} + a\sqrt{q}}{p - q}.$$

If one of the terms of the denominator is rational, we may make the denominator rational by the same method.

EXAMPLES.

Make rational the denominators of the following:

1. $\frac{7}{3 - \sqrt{5}}$. *Ans.* $\frac{21 + 7\sqrt{5}}{4} = 9$, to within $\frac{1}{4}$.

2. $\frac{7\sqrt{5}}{\sqrt{11} + \sqrt{3}}$. *Ans.* $\frac{7\sqrt{55} - 7\sqrt{15}}{8} = 3.1$ to within $\frac{1}{800}$.

3. $\frac{3 + 2\sqrt{7}}{5\sqrt{12} - 6\sqrt{5}}$.

Ans. $\frac{(3 + 2\sqrt{7})(5\sqrt{12} + 6\sqrt{5})}{120} = 2.123$ to within $\frac{1}{1000}$.

208 [illegible] St.
San Francisco
California.

Simplify the following:

4. $\dfrac{(3+\sqrt{3})(3+\sqrt{5})(\sqrt{5}-2)}{(5-\sqrt{5})(1+\sqrt{3})}$. *Ans.* $\frac{1}{5}\sqrt{15}$.

5. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$. *Ans.* $\dfrac{a+\sqrt{a^2-x^2}}{x}$.

6. $\dfrac{(7-2\sqrt{5})+2+\sqrt{45}}{2\sqrt{3}-\sqrt{7}}$. *Ans.* $\dfrac{(9+\sqrt{5})(2\sqrt{3}+\sqrt{7})}{5}$.

1 — 6 - 78

MISCELLANEOUS EXAMPLES.

1. Simplify $\sqrt{125}$. *Ans.* $5\sqrt{5}$.

2. Reduce $\sqrt{\dfrac{50}{147}}$ to its simplest form.

We observe that 25 will divide the numerator, and hence,

$$\sqrt{\frac{50}{147}}=\sqrt{\frac{25\times 2}{147}}=5\sqrt{\frac{2}{147}}.$$

Since the perfect square 49 will divide 147,

$$5\sqrt{\frac{2}{147}}=5\sqrt{\frac{2}{49\times 3}}=\frac{5}{7}\sqrt{\frac{2}{3}}.$$

Divide the coefficient of the radical by 3, and multiply the number under the radical by the square of 3; then,

$$\frac{5}{7}\sqrt{\frac{2}{3}}=\frac{5}{21}\sqrt{\frac{18}{3}}=\frac{5}{21}\sqrt{6}. \quad \textit{Ans.}$$

3. Reduce $\sqrt{98a^2x}$ to its most simple form. *Ans.* $7a\sqrt{2x}$.

4. Reduce $\sqrt{(x^3-a^2x^2)}$ to its most simple form. *Ans.* $x\sqrt{x-a^2}$.

5. Required the sum of $\sqrt{72}$ and $\sqrt{128}$. *A.* $14\sqrt{2}$.

6. Required the sum of $\sqrt{27}$ and $\sqrt{147}$.

Ans. $10\sqrt{3}$.

7. Required the sum of $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{27}{50}}$

Ans. $\frac{19}{30}\sqrt{6}$.

8. Required the sum of $2\sqrt{a^2b}$ and $3\sqrt{64bx^4}$.

Ans. $(2a + 24x^2)\sqrt{b}$.

9. Required the sum of $9\sqrt{243}$ and $10\sqrt{363}$.

Ans. $191\sqrt{3}$.

10. Required the difference of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$.

Ans. $\frac{4}{45}\sqrt{15}$.

11. Required the product of $5\sqrt{8}$ and $3\sqrt{5}$.

Ans. $30\sqrt{10}$.

12. Find the product of $\frac{2}{3}\sqrt{\frac{1}{8}}$ and $\frac{3}{4}\sqrt{\frac{7}{10}}$.

Ans. $\frac{1}{40}\sqrt{35}$.

13. Divide $6\sqrt{10}$ by $3\sqrt{5}$. *Ans.* $2\sqrt{2}$.

14. Find the sum of $\sqrt{18a^5b^3}$ and $\sqrt{50a^3b^3}$.

Ans. $(3a^2b + 5ab)\sqrt{2ab}$.

15. Find the algebraic sum of

$$4\sqrt{\frac{7}{3}},\quad -5\sqrt{\frac{1}{2}},\quad \text{and}\quad 12\sqrt{\frac{3}{4}}.$$

Ans. $\dfrac{8\sqrt{21} - 15\sqrt{2} + 36\sqrt{3}}{6}$.

CHAPTER VI.

EQUATIONS OF THE SECOND DEGREE.

I. EQUATIONS WITH ONE UNKNOWN QUANTITY.

119. Equations of the second degree may contain *but one* unknown quantity, or they may contain *more than one.* The former class will be considered first.

Equations of this class are often called *Quadratic Equations.*

120. Degree.—An equation containing but one unknown quantity is of the *second degree* when the greatest exponent of the unknown quantity, in any term, is 2.

Simplest Form.—To deduce a rule for reducing such an equation to its simplest form, let us take the equation,

$$\frac{a}{3}x^2 - cx - 4 = \frac{2}{5}x^2 - \frac{4}{3}.$$

Clearing of fractions,

$$5ax^2 - 15cx - 60 = 6x^2 - 20.$$

Transposing and arranging,

$$5ax^2 - 6x^2 - 15cx = +60 - 20.$$

Factoring and reducing,

$$(5a - 6)x^2 - 15cx = 40.$$

Dividing both members by the coefficient of x^2, we have,

$$x^2 - \frac{15c}{5a-6}x = \frac{40}{5a-6}.$$

If we now replace the coefficient of x by $2p$, and the second member by q, we shall have,

$$x^2 + 2px = q;$$

and since all similar equations may be reduced in like manner, we conclude that every equation of the second degree, containing but one unknown quantity, can be reduced to the form

$$x^2 + 2px = q,$$

by the following

RULE.

I. Clear the equation of fractions:

II. Transpose all the known terms to the second member, and all the unknown terms to the first:

III. Reduce the terms involving the square of the unknown quantity to a single term of two factors, one of which is the square of the unknown quantity:

IV. Then, divide both members by the coefficient of the square of the unknown quantity.

121. When $2p = 0$, that is, when the coefficient of the second term of the reduced equation is equal to 0, that equation is said to be *incomplete.* When $2p$ is not equal to 0, the equation is *complete.*

The incomplete equation is often called a *Pure Quadratic,* and the complete equation is called an *Adfected Quadratic.*

122. Incomplete Equations.—An *incomplete* equation can always be reduced to the form

$$x^2 = q,$$

by the rule of Article **120.** If we extract the square root of both members of this equation, remembering that every quantity has two square roots (Art. **111**), we have,

$$x = \pm \sqrt{q}.$$

In this case, the equation has two roots, numerically equal, but having contrary signs.

Denoting the first root by x' and the second by x'', we have,

$$x' = +\sqrt{q} \qquad \text{and} \qquad x'' = -\sqrt{q}.$$

Hence, for the solution of incomplete equations, we have the following

RULE.

Reduce the equation to the form $x^2 = q$; *then extract the square root of both members; the first root of the equation is equal to the positive value of the square root of the second member of the equation, and the second root is equal to the negative value.*

EXAMPLES.

Solve the following equations:

1. $\frac{x^2}{3} - 3 + \frac{5}{12}x^2 = \frac{7}{24} - x^2 + \frac{299}{24}.$

Clearing of fractions,

$$8x^2 - 72 + 10x^2 = 7 - 24x^2 + 299;$$

Transposing and reducing,

$$42x^2 = 378; \qquad \therefore \qquad x^2 = 9.$$

Extracting the square root of both members,

$$x' = +3 \qquad \text{and} \qquad x'' = -3.$$

2. $3x^2 = 5.$ *Ans.* $x' = +\frac{1}{3}\sqrt{15},\ x'' = -\frac{1}{3}\sqrt{15}.$

3. $11(x^2 - 4) = 5(x^2 + 2).$ *Ans.* $x' = +3,\ x'' = -3.$

4. $\dfrac{\sqrt{m^2 - x^2}}{x} = m.$

Ans. $x' = + \dfrac{m}{\sqrt{1+m^2}},\quad x'' = - \dfrac{m}{\sqrt{1+m^2}}.$

5. $9x^2 + 9 = 3x^2 + 63.$ *Ans.* $x' = +3,\quad x'' = -3.$

6. $8\left(x + \dfrac{1}{x}\right) = \dfrac{21 + 70x}{7} - 3.$ *Ans.* $x' = +2,\quad x'' = -2.$

7. $\dfrac{3}{1+x} + \dfrac{3}{1-x} = 8.$ *Ans.* $x' = +\dfrac{1}{2},\quad x'' = -\dfrac{1}{2}.$

8. $\dfrac{x+4}{x-4} + \dfrac{x-4}{x+4} = \dfrac{10}{3}.$ *Ans.* $x' = +8,\quad x'' = -8.$

9. $\dfrac{x-1}{x} - \dfrac{3x}{x-1} = 2.$ *Ans.* $x' = +\dfrac{1}{2},\quad x'' = -\dfrac{1}{2}.$

10. $\dfrac{x}{12} + \dfrac{x^2 - 15}{5x} = \dfrac{x}{5}.$ *Ans.* $x' = +6,\quad x'' = -6.$

123. Complete Equations.—The complete equation of the second degree may be reduced to the form

$$x^2 + 2px = q.$$

If we compare the first member of this equation with the square of

$$x + p, \qquad \text{which is} \qquad x^2 + 2px + p^2,$$

we see that it needs but the square of p to render it a perfect square. If then, p^2 be added to the first member, it will become a perfect square; but to preserve the equality of the members, p^2 must also be added to the second member. Making these additions, we have

$$x^2 + 2px + p^2 = q + p^2;$$

this is called, *completing the square*, and is done, *by adding the square of half the coefficient of x to both members of the equation.*

Now, if we extract the square root of both members, we have

$$x + p = \pm\sqrt{q + p^2}.$$

Transposing p to the second member, we have

$$x = -p \pm \sqrt{q + p^2}.$$

If we denote the value of x, corresponding to the upper sign of the radical, by x', and that corresponding to the lower sign by x'', we have for the two roots

$$x' = -p + \sqrt{q + p^2} \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

$$x'' = -p - \sqrt{q + p^2} \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

Either of these values, substituted for x in the general equation,

$$x^2 + 2px = q,$$

will satisfy it. The one corresponding to the + sign before the radical is called the *first* root, and the other, the *second* root of the equation. The preceding method of solution may be applied to all cases of complete equations of the second degree.

EXAMPLES.

Solve the following equations:

1. $\dfrac{x}{7} + \dfrac{21}{x + 5} = \dfrac{23}{7}.$

Clearing of fractions,

$$x^2 + 5x + 147 = 23x + 115.$$

Transposing and reducing,

$$x^2 - 18x = -32.$$

Completing the square,

$$x^2 - 18x + 81 = 49.$$

Extracting the square root of both members,

$$x - 9 = \pm 7;$$

$$\therefore \qquad x' = 16 \qquad \text{and} \qquad x'' = +2.$$

2. $\dfrac{21}{5 - x} - \dfrac{x}{7} = \dfrac{23}{7}.$

Clearing of fractions,

$$147 - 5x + x^2 = 115 - 23x.$$

Transposing and reducing,

$$x^2 + 18x = -32.$$

Completing the square,

$$x^2 + 18x + 81 = 49.$$

Extracting the square root,

$$x + 9 = \pm 7;$$

$$\therefore \; x' = -9 + 7 = -2, \quad \text{and} \quad x'' = -9 - 7 = -16.$$

In like manner all similar examples may be solved: but a simpler method consists in reducing the given equation to the form

$$x^2 + 2px = q,$$

and then applying the following rule deduced from formulas (1) and (2),

RULE.

I. The first root is equal to half the coefficient of the second term, taken with a contrary sign, plus the square root of the second member increased by the square of half the coefficient of the second term:

II. The second root is equal to half the coefficient of the second term, taken with a contrary sign, minus the square root of the second member increased by the square of half the coefficient of the second term.

ADDITIONAL EXAMPLES.

3. $x^2 - 7x + 10 = 0.$

Reducing to the required form,

$$x^2 - 7x = -10.$$

By the rule,

$$x' = \frac{7}{2} + \sqrt{-10 + \frac{49}{4}} = 5; \qquad \text{and}$$

$$x'' = \frac{7}{2} - \sqrt{-10 + \frac{49}{4}} = 2.$$

4. $\frac{5}{6}x^2 - \frac{1}{2}x + \frac{3}{4} = 8 - \frac{2}{3}x - x^2 + \frac{273}{10}\checkmark$

Reducing to the required form,

$$x^2 + \frac{1}{11}x = \frac{180}{11}.$$

By the rule,

$$x' = -\frac{1}{22} + \sqrt{\frac{180}{11} + \frac{1}{484}} = 4; \qquad \text{and,}$$

$$x'' = -\frac{1}{22} - \sqrt{\frac{180}{11} + \frac{1}{484}} = -\frac{45}{11}.$$

When possible, *reduce the quantities under the radic sign to a common denominator, which shall be a perf square; then add or subtract the numerators, and reduce th result to its simplest form.*

5. $6x^2 - 37x + 57 = 0.$ *Ans.* $x' = 3\frac{1}{6}, \quad x'_{,} = 3.$

6. $4a^2 - 2x^2 + 2ax = 18ab - 18b^2.$
Ans. $x' = 2a - 3b, \quad x'' = -a + 3b.$

7. $\frac{x^2}{3} - \frac{a}{b}x = 1 - \frac{b}{a}x - \frac{2x^2}{3}.$ *Ans.* $x' = \frac{a}{b}, \quad x'' = -\frac{b}{a}.$

8. $\frac{dx}{c} + \frac{3x^2}{4} + 1 = \frac{1+c}{c} - \frac{x^2}{4} + \frac{x}{d}.$
Ans. $x' = \frac{1}{d}, \quad x'' = -\frac{d}{c}.$

9. $\frac{x^2}{4} - \frac{2x}{3} + \frac{59}{8} = 8 - \frac{x^2}{4} - \frac{x}{3}$.

Ans. $x' = \frac{3}{2}$, $x'' = -\frac{5}{6}$.

10. $\frac{90}{x} - \frac{90}{x+1} = \frac{27}{x+2}$. *Ans.* $x' = 4$, $x'' = -\frac{5}{3}$.

11. $\frac{2x-10}{8-x} - 2 = \frac{x+3}{x-2}$. *Ans.* $x' = 7$, $x'' = \frac{4}{5}$.

12. $ax - \frac{x^2}{b} + b = \frac{b-1}{b}x^2 + \frac{b}{a}x$.

Ans. $x' = a$, $x'' = -\frac{b}{a}$.

13. $\frac{a-b}{c}x + \frac{3x^2}{2} - \frac{a^2}{c^2} = \frac{b+a}{c}x + \frac{x^2}{2} - \frac{b^2}{c^2}$.

Ans. $x' = \frac{b+a}{c}$, $x'' = \frac{b-a}{c}$.

14. $mx^2 + mn = 2m\sqrt{n}\,x + nx^2$.

Ans. $x = \frac{\sqrt{mn}}{\sqrt{m} - \sqrt{n}}$, $x'' = \frac{\sqrt{mn}}{\sqrt{m} + \sqrt{n}}$.

15. $abx^2 - \frac{6a^2}{c^2} + \frac{b^2x}{c} = \frac{ab - 2b^2}{c^2} - \frac{3a^2}{c}x$.

Ans. $x' = \frac{2a-b}{ac}$, $x'' = -\frac{3a+2b}{bc}$.

16. $\frac{4x^2}{7} + \frac{2x}{7} + 10 = 19 - \frac{3x^2}{7} + \frac{58x}{7}$.

Ans. $x' = 9$, $x'' = -1$.

17. $\frac{x+a}{x-a} - b = \frac{a-x}{a+x}$. *Ans.* $x = \pm a\sqrt{\frac{b+2}{b-2}}$.

18. $2x + 2 = 24 - 5x - 2x^2$ *Ans.* $x' = 2$, $x'' = -\frac{11}{2}$.

19. $x^2 - x - 40 = 170$. *Ans.* $x' = 15$, and $x'' = -14$.

20. $3x^2 + 2x - 9 = 76$. *Ans.* $x' = 5$, and $x'' = -5\frac{2}{3}$.

21. $a^2 + b^2 - 2bx + x^2 = \frac{m^2x^2}{n^2}$.

Ans. $x = \frac{n}{n^2 - m^2}\left(bn \pm \sqrt{a^2m^2 + b^2m^2 - a^2n^2}\right)$.

ADDITIONAL EXAMPLES.

3. $x^2 - 7x + 10 = 0.$

Reducing to the required form,

$$x^2 - 7x = -10.$$

By the rule,

$$x' = \frac{7}{2} + \sqrt{-10 + \frac{49}{4}} = 5; \qquad \text{and}$$

$$x'' = \frac{7}{2} - \sqrt{-10 + \frac{49}{4}} = 2.$$

4. $\frac{5}{6}x^2 - \frac{1}{2}x + \frac{3}{4} = 8 - \frac{2}{3}x - x^2 + \frac{273}{10}$ ✓

Reducing to the required form,

$$x^2 + \frac{1}{11}x = \frac{180}{11}.$$

By the rule,

$$x' = -\frac{1}{22} + \sqrt{\frac{180}{11} + \frac{1}{484}} = 4; \qquad \text{and,}$$

$$x'' = -\frac{1}{22} - \sqrt{\frac{180}{11} + \frac{1}{484}} = -\frac{45}{11}.$$

When possible, *reduce the quantities under the radical sign to a common denominator, which shall be a perfect square; then add or subtract the numerators, and reduce the result to its simplest form.*

5. $6x^2 - 37x + 57 = 0.$ *Ans.* $x' = 3\frac{1}{6},\ x'_{,} = 3.$

6. $4a^2 - 2x^2 + 2ax = 18ab - 18b^2.$
Ans. $x' = 2a - 3b, \quad x'' = -a + 3b.$

7. $\frac{x^2}{3} - \frac{a}{b}x = 1 - \frac{b}{a}x - \frac{2x^2}{3}.$ *Ans.* $x' = \frac{a}{b}, \quad x'' = -\frac{b}{a}.$

8. $\frac{dx}{c} + \frac{3x^2}{4} + 1 = \frac{1+c}{c} - \frac{x^2}{4} + \frac{x}{d}.$
Ans. $x' = \frac{1}{d}, \quad x'' = -\frac{d}{c}.$

9. $\frac{x^2}{4} - \frac{2x}{3} + \frac{59}{8} = 8 - \frac{x^2}{4} - \frac{x}{3}$.

Ans. $x' = \frac{3}{2}$, $x'' = -\frac{5}{6}$.

10. $\frac{90}{x} - \frac{90}{x+1} = \frac{27}{x+2}$. *Ans.* $x' = 4$, $x'' = -\frac{5}{3}$.

11. $\frac{2x-10}{8-x} - 2 = \frac{x+3}{x-2}$. *Ans.* $x' = 7$, $x'' = \frac{4}{5}$.

12. $ax - \frac{x^2}{b} + b = \frac{b-1}{b}x^2 + \frac{b}{a}x$.

Ans. $x' = a$, $x'' = -\frac{b}{a}$.

13. $\frac{a-b}{c}x + \frac{3x^2}{2} - \frac{a^2}{c^2} = \frac{b+a}{c}x + \frac{x^2}{2} - \frac{b^2}{c^2}$.

Ans. $x' = \frac{b+a}{c}$, $x'' = \frac{b-a}{c}$.

14. $mx^2 + mn = 2m\sqrt{n}\,x + nx^2$.

Ans. $x = \frac{\sqrt{mn}}{\sqrt{m} - \sqrt{n}}$, $x'' = \frac{\sqrt{mn}}{\sqrt{m} + \sqrt{n}}$.

15. $abx^2 - \frac{6a^2}{c^2} + \frac{b^2x}{c} = \frac{ab - 2b^2}{c^2} - \frac{3a^2}{c}x$.

Ans. $x' = \frac{2a-b}{ac}$, $x'' = -\frac{3a+2b}{bc}$.

16. $\frac{4x^2}{7} + \frac{2x}{7} + 10 = 19 - \frac{3x^2}{7} + \frac{58x}{7}$.

Ans. $x' = 9$, $x'' = -1$.

17. $\frac{x+a}{x-a} - b = \frac{a-x}{a+x}$. *Ans.* $x = \pm a\sqrt{\frac{b+2}{b-2}}$.

18. $2x + 2 = 24 - 5x - 2x^2$ *Ans.* $x' = 2$, $x'' = -\frac{11}{2}$.

19. $x^2 - x - 40 = 170$. *Ans.* $x' = 15$, and $x'' = -14$.

20. $3x^2 + 2x - 9 = 76$. *Ans.* $x' = 5$, and $x'' = -5\frac{2}{3}$.

21. $a^2 + b^2 - 2bx + x^2 = \frac{m^2x^2}{n^2}$.

Ans. $x = \frac{n}{n^2 - m^2}\left(bn \pm \sqrt{a^2m^2 + b^2m^2 - a^2n^2}\right)$.

guage is more general in its character than that used in enunciating the problem. As a consequence, we may obtain more solutions than will satisfy the conditions of the problem; in this case, the superfluous ones are to be rejected.

3. A man bought a horse, which he sold for 24 dollars. At the sale, he lost as much per cent. on the price of his purchase as the horse cost him. What did he pay for the horse?

Let x denote the number of dollars that he paid for the horse; then, $x - 24$ will denote the number of dollars that he lost.

But as he lost x per cent. by the sale, he must have lost $\frac{x}{100}$ upon each dollar, and upon x dollars he lost a number of dollars denoted by $\frac{x^2}{100}$; we have then the equation

$$\frac{x^2}{100} = x - 24; \quad \text{whence,} \quad x^2 - 100x = -2400;$$

Therefore, $x' = 60$ and $x'' = 40$.

Both these values satisfy the conditions of the problem.

For, if he gave 60 dollars, he must have lost $\$60 \times \frac{60}{100}$, or \$36, in which case he would receive \$24.

If he gave 40 dollars, he must have lost $\$40 \times \frac{40}{100}$, or \$16, in which case he would receive \$24.

4. A grazier bought as many sheep as cost him £60, and after reserving 15 of the number, he sold the remainder for £54, and gained 2*s.* a head on those he sold: how many did he buy? *Ans.* 75.

5. A merchant bought several pieces of cloth for which he paid £33 15*s.*, and sold them again at £2 8*s.* per piece, gaining by the bargain as much as one piece cost him: how many pieces did he buy? *Ans.* 15.

6. What number is that, which, being divided by the product of its digits, the quotient will be 3; and if 18 be added to it, the order of its digits will be reversed? *Ans.* 24.

7. Find a number such that if you subtract it from 10, and multiply the remainder by the number itself, the product will be 21. *Ans.* 7 or 3.

8. Two persons, A and B, departed from different places at the same time, and traveled toward each other. On meeting, it appeared that A had traveled 18 miles more than B; and that A could have performed B's journey in $15\frac{3}{4}$ days, but B would have been 28 days in performing A's journey. How far did each travel? *Ans.* $\begin{cases} \text{A, 72 miles.} \\ \text{B, 54 miles.} \end{cases}$

9. A company at a tavern had £8 15*s.* to pay for their reckoning; but before the bill was settled, two of them left the room, and then those who remained had 10*s.* apiece more to pay than before: how many were there in the company? *Ans.* 7.

10. What two numbers are those whose difference is 15, and of which the cube of the less is equal to half their product? *Ans.* 3 and 18.

11. Two partners, A and B, gained $140 in trade: A's money was three months in trade, and his gain was $60 less than his stock; B's money was $50 more than A's, and was in trade five months. What was A's stock? *Ans.* $100.

12. Two persons, A and B, start from different points, and travel toward each other. When they meet, it appears that A has traveled 30 miles more than B. It also appears that it will take A four days to travel the road that B has come, and B, nine days to travel the road that A has come. What was their distance apart when they set out? *Ans.* 150 miles.

III. PROPERTIES OF THE EQUATION OF THE SECOND DEGREE.

124. It has been shown that every complete equation of the second degree can be reduced to the form

$$x^2 + 2px = q \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

in which p and q are *numerical* or *literal, positive* or *negative.*

If we add p^2 to both members, and then transpose the second member to the first, we have

$$(x^2 + 2px + p^2) - (q + p^2) = 0,$$

or,

$$(x + p)^2 - (\sqrt{q + p^2})^2 = 0 \quad . \; . \; . \; . \; . \quad (2)$$

Factoring the first member of equation (2) according to the principle that *the difference of the squares of two quantities, is equal to the sum of the quantities multiplied by their difference,* we have

$$(x + p + \sqrt{q + p^2})\,(x + p - \sqrt{q + p^2}) = 0 \quad . \; . \quad (3)$$

Equation (3) can be satisfied by making each of the factors of its first member equal to 0, and it can be satisfied in no other way.

Making these factors separately equal to 0, solving and denoting the corresponding values of x by x' and x'', we have

$$x + p - \sqrt{q + p^2} = 0, \quad \therefore \quad x' = -p + \sqrt{q + p^2} \; . \; . \; (4)$$

$$x + p + \sqrt{q + p^2} = 0, \quad \therefore \quad x'' = -p - \sqrt{q + p^2} \; . \; . \; (5)$$

These values of x' and x'' are roots of equation (1); and if we substitute them in equation (3), we have

$$(x - x'')\,(x - x') = 0 \quad . \; . \; . \; . \; . \; . \quad (6)$$

Hence, we conclude,

1°. *Every equation of the second degree has two roots, and only two.*

2°. *The first member of every equation of the second degree whose second member is 0, can be resolved into two binomial factors, having the unknown quantity for a first term, and the two roots, with their signs changed, for second terms.*

The second principle enables us to form an equation of the second degree when its roots are given. To do this, we subtract each root from the unknown quantity, separately, and then multiply the results together.

EXAMPLES.

1. What are the factors, and what is the equation, of which the roots are 8 and -9?

Ans. $x-8$ and $x+9$ are the binomial factors,
and $x^2+x-72=0$ is the equation.

2. What are the factors, and what is the equation, of which the roots are -1 and $+1$?

Ans. $x+1$ and $x-1$ are the factors,
and $x^2-1=0$ is the equation.

3. What are the factors, and what is the equation, whose roots are

$$\frac{7+\sqrt{-1039}}{16} \text{ and } \frac{7-\sqrt{-1039}}{16}?$$

Ans. $\left(x-\frac{7+\sqrt{-1039}}{16}\right)$ and $\left(x-\frac{7-\sqrt{-1039}}{16}\right)$

are the factors,

and $8x^2-7x+34=0$ is the equation.

If we add equations (4) and (5), member to member, and reduce, we have

$$x'+x''=-2p \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

If we multiply equations (4) and (5) together, member by member, and reduce, we have

$$x'x'' = -q \quad \ldots\ldots\ldots \quad (8)$$

Hence, after an equation has been reduced to the form of

$$x^2 + 2px = q,$$

3°. *The algebraic sum of its two roots is equal to the coefficient of the second term, with its sign changed.*

4°. *The product of the two roots is equal to the second member, with its sign changed.*

It follows from the last principle that the product of the roots can only be + when the second member is −; but the product can only be + when their signs are alike. In this case, the sum of the two roots is $-2p$; and if we denote their difference by $2d$, the two roots will be equal to

$$-p + d \text{ and } -p - d; \text{ and}$$

Multiplying these together, we have,

$$p^2 - d^2.$$

This product is greatest possible when $d = 0$; that is, when the roots are equal. In all other cases, the product of the roots is less than p^2. Hence,

5°. *When the second member is negative, it can never be numerically greater than the square of half the coefficient of the second term.*

These principles, demonstrated in this and the preceding article, are employed in the discussion of the general equation of the second degree.

125. Discussion of the Four Forms.—The *discussion of an equation* consists in making every possible supposition on the arbitrary quantities that enter it, and then interpreting the results.

In the reduced equation,

$$x^2 + 2px = q,$$

the quantities p and q are arbitrary. By assigning suitable values to them, this equation may be made to represent every equation of the second degree.

To discuss this equation, it will be convenient, in the first place, to make every hypothesis on the *signs* of p and q, and afterwards to consider their *relative numerical values.*

We can suppose that both p and q are essentially positive; that p is negative and q positive; that p is positive and q negative; or, that both p and q are negative. Hence, the reduced equation may take any one of the *four* following forms:

$$x^2 + 2px = q \quad \ldots\ldots\ldots \quad (1)$$
$$x^2 - 2px = q \quad \ldots\ldots\ldots \quad (2)$$
$$x^2 + 2px = -q \quad \ldots\ldots\ldots \quad (3)$$
$$x^2 - 2px = -q \quad \ldots\ldots\ldots \quad (4)$$

These equations being solved, give

$$x = -p \pm \sqrt{q + p^2} \quad \ldots\ldots \quad (1)$$
$$x = +p \pm \sqrt{q + p^2} \quad \ldots\ldots \quad (2)$$
$$x = -p \pm \sqrt{-q + p^2} \quad \ldots\ldots \quad (3)$$
$$x = +p \pm \sqrt{-q + p^2} \quad \ldots\ldots \quad (4)$$

1°. In the first and second forms, the quantity under the radical sign will be positive, whatever be the relative values of p and q, since q and p^2 are both positive; therefore, both roots are real. And since

$$q + p^2 > p^2, \text{ it follows that, } \sqrt{q + p^2} > p;$$

consequently, *the roots in both these forms have the same signs as the radicals.*

In the *first* form, the first root is positive and the second negative, the negative root being numerically the greater.

In the *second* form, the first root is positive and the second negative, the positive root being numerically the greater.

2°. In the third and fourth forms, if

$$p^2 > q,$$

the roots will be real, and since

$$p > \sqrt{-q + p^2},$$

they have the same sign as p. Hence, *both roots are negative in the third form, and both positive in the fourth.*

3°. If $p^2 = q$, the quantity under the radical sign in the third and fourth forms is 0, and the two values of x are equal; both equal to $-p$ in the third form, and both equal to $+p$ in the fourth.

4°. If $p^2 < q$, the quantity under the radical sign in the third and fourth forms is negative, and all the roots in these forms are imaginary.

But in this case, the second member in both is negative, and according to the fifth principle of Article **124**, the value of q can never be greater than p^2. Hence, the hypothesis that makes the roots imaginary is *absurd;* the roots ought, therefore, to be *impossible* or *imaginary.*

When any supposition gives rise to imaginary roots, we interpret the result as indicating that the supposition is *impossible* or *absurd.*

5°. If p is equal to 0, and q is not equal to 0, the roots of the first and second forms reduce to $\pm\sqrt{q}$, and those of the third and fourth forms reduce to $\pm\sqrt{-q}$; that is, the two roots in each case are equal with contrary signs, being *real* in the former and imaginary in the latter case.

6°. If q is equal to 0, and p is not equal to 0, one root in each form is 0, and the other is equal to the coefficient of the second term with its sign changed.

7°. If both p and q are equal to 0, both roots in each of the four forms reduces to 0.

These suppositions embrace every possible hypothesis that can be made upon p and q.

126. The results deduced in Article **125** might have been obtained by a discussion of the *four forms* themselves, instead of their roots, making use of the principles demonstrated in Article **124.**

1°. In the *first* form the product of the two roots is equal to $-q$; hence, the roots must have contrary signs; their sum is $-2p$; hence, the negative root is numerically the greater.

2°. In the *second* form the product of the roots is equal to $-q$; hence, their signs are unlike; their sum is equal to $+2p$; hence, the positive root is the greater.

3°. In the *third* form the product of the roots is equal to $+q$; hence, their signs are alike; and their sum being equal to $-2p$, they are both negative.

4°. In the *fourth* form the product of the roots is equal to $+q$; hence, their signs are alike; their sum is equal to $+2p$; hence, both are positive.

5°. If $p = 0$, the sum of the roots must be equal to 0; hence, the roots must be equal with contrary signs.

6°. If $q = 0$, the product of the roots is 0; hence, one of them must be 0, and the other equal to the coefficient of the second term with its sign changed.

7°. If $p = 0$ and $q = 0$, the sum of the roots is 0, and their product is 0; hence, both the roots are equal to 0.

127. There is a case, sometimes met with, in the discussion of equations of the second degree, that requires explanation. To discuss it, let us take the equation,

$$ax^2 + bx = c.$$

Solving this equation and denoting its roots by x' and x'', we have,

$$x' = \frac{-b + \sqrt{b^2 + 4ac}}{2a}, \text{ and } x'' = \frac{-b - \sqrt{b^2 + 4ac}}{2a}.$$

If we suppose $a = 0$, the two roots become,

$$x' = \frac{-b + b}{0} = \frac{0}{0} \text{ and } x'' = \frac{-b - b}{0} = \infty.$$

But the supposition reduces the given equation to the form

$$bx = c, \qquad \text{whence} \qquad x' = \frac{c}{b}.$$

To ascertain the true value of the first root, we multiply both numerator and denominator by $b + \sqrt{b^2 + 4ac}$, which gives, after reduction,

$$x' = \frac{b^2 - (b^2 + 4ac)}{2a(b + \sqrt{b^2 + 4ac})} = \frac{4ac}{2a(b + \sqrt{b^2 + 4ac})}.$$

Striking out the common factor $2a$, which reduces to 0, under the particular hypothesis (Art. **80**), we have

$$x' = \frac{2c}{b + \sqrt{b^2 + 4ac}}.$$

Making $a = 0$, in this result we find, as before,

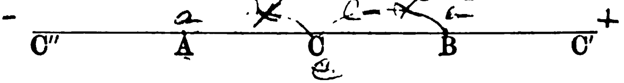
$$x' = \frac{c}{b}.$$

128. Discussion of Problems.—The problem of the lights offers an illustration of the general method of discussing a problem of the second degree. This problem depends on the following principle of physics:

The intensity of a light at any distance, is equal to its intensity at the distance 1, *divided by the square of that distance.*

Problem.—Find, on the line that joins two lights, A and B, of different intensities, a point that is equally illuminated by both lights.

129. Solution.—Let A and B represent the places of the two lights, and let A be taken as the origin of distances. Call all distances from A toward the right *positive;* then will all distances toward the left be negative. Let c denote the distance AB, between the lights, a the intensity of the light A, at the distance 1, and b the intensity of the light B, at the same distance. Suppose C to be a point on AB, which is equally illuminated, and denote the distance AC by x; then will BC be denoted by $c - x$.



From the assumed principle of physics, the intensity of the first light at C, is equal to $\frac{a}{x^2}$, and the intensity of the second light at the same point, is $\frac{b}{(c-x)^2}$; hence, from the conditions of the problem, we have,

$$\frac{a}{x^2} = \frac{b}{(c-x)^2}.$$

Extracting the square root of both members, we have

$$\frac{\sqrt{a}}{x} = \pm\frac{\sqrt{b}}{c-x}.$$

Taking the upper sign, and denoting the corresponding value of x by x', we have

$$x' = \frac{c\sqrt{a}}{\sqrt{a}+\sqrt{b}}. \quad \dots\dots\dots \quad (1)$$

Taking the lower sign, and denoting the corresponding value of x by x'', we have

$$x'' = \frac{c\sqrt{a}}{\sqrt{a}-\sqrt{b}}. \quad \dots\dots\dots \quad (2)$$

130. Discussion.—From the nature of the case, a, b, and c are positive; hence, both values of x are *real.* This shows that there are two points of the line that are equally illuminated by the two lights.

Since there must be two lights, it is impossible that either a, b, or c should be equal to 0. Hence, there are but three hypotheses that can be made.

$$\begin{array}{llll} 1^\circ. & c > 0 & \text{and} & a > b; \\ 2^\circ. & c > 0 & \text{and} & a < b; \\ 3^\circ. & c > 0 & \text{and} & a = b. \end{array}$$

1°. *Suppose* $c > 0$ *and* $a > b$.

In this case the value of x' is *positive;* and since $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$ is a proper fraction, the value of x' is less than c. Hence, the first point of equal illumination lies between A and B. We see, furthermore, that the denominator of the fraction $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$ is less than twice the numerator; hence, $x' > \frac{1}{2}c$. This shows that the first point of equal illumination is nearer the less light.

The value of x'' is also *positive,* but $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$ is an improper fraction; hence, the second point C′ of equal illumination lies to the right of both lights; that is, it lies on the prolongation of the line joining them, and on the side of the less light.

These conclusions are as they should be; for, since the light A is the more intense, the point of equal illumination, between the two lights, ought to be nearer the light B; also, the point without both, ought to be on the side of the lesser light.

2°. *Suppose* $c > 0$ *and* $a < b$.

In this case the value of x' is *positive,* and for the same ason as before it is less than c. Hence, the first point of equal illumination is between the two lights. We see, furthermore, that the denominator of the fraction $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$ is greater than twice the numerator; hence, $x' < \frac{1}{2}c$. This shows that the first point of equal illumination is nearer the less light.

The value of x'' is *negative*. This shows that the second point of equal illumination lies to the left of A; that is, the point C″, of equal illumination, is on the prolongation of the line joining them, and on the side of the less light.

These conclusions are also as they should be; for, in this case the light B is the more intense, and for the same reason as before, the point of equal illumination between them ought to be nearer the light A; also, the point without both ought to be on the side of the feebler light.

3°. *Suppose* $c > 0$ *and* $a = b$.

In this case the value of x' becomes equal to $\frac{1}{2}c$, and the value of x'' is equal to ∞. Hence, we infer that there is a point of equal illumination midway between the two lights, and that there is no other point of equal illumination on the line joining the lights.

These conclusions are as they should be; for, in this case the two lights are of equal intensity, and the point midway between ought to be equally illuminated. It is equally obvious that there can be no other point, at a finite distance from A, on the line of the lights, or on that line prolonged, that is equally illuminated.

We have said that the conditions of the problem require two lights, and the equation of the problem has been deduced on this hypothesis. This equation ought not to respond to the case of a single light; for the hypothesis of a single light and the hypothesis of two lights, are not connected by any law that can give a common equation. Hence, the results obtained by making $c = 0$ do not belong to the problem under discussion.

131. Radical Equations.—Many equations involving radicals can be reduced to the form

$$x^2 + 2px = q,$$

and then solved by the methods already given.

EXAMPLES.

1. Given $x + \sqrt{a^2 + x^2} = \dfrac{2a^2}{\sqrt{a^2 + x^2}}.$

Clearing of fractions,

$$x\sqrt{a^2 + x^2} + a^2 + x^2 = 2a^2.$$

Transposing,

$$x\sqrt{a^2 + x^2} = a^2 - x^2.$$

Squaring both members,

$$x^2(a^2 + x^2) = a^4 - 2a^2x^2 + x^4.$$

Transposing and reducing,

$$3a^2x^2 = a^4;$$

$$\therefore \quad x' = +\frac{a}{\sqrt{3}} \text{ and } x'' = -\frac{a}{\sqrt{3}}.$$

2. Given $\sqrt{\dfrac{a^2}{x^2} + b^2} - \sqrt{\dfrac{a^2}{x^2} - b^2} = b.$

Transposing,

$$\sqrt{\frac{a^2}{x^2} + b^2} = b + \sqrt{\frac{a^2}{x^2} - b^2}.$$

Squaring both members,

$$\frac{a^2}{x^2} + b^2 = b^2 + 2b\sqrt{\frac{a^2}{x^2} - b^2} + \frac{a^2}{x^2} - b^2.$$

Transposing and reducing,

$$b = 2\sqrt{\frac{a^2}{x^2} - b^2}.$$

Squaring both members,

$$b^2 = 4\left(\frac{a^2}{x^2} - b^2\right);$$

$$\therefore \quad x' = +\frac{2a}{b\sqrt{5}} \quad \text{and} \quad x'' = -\frac{2a}{b\sqrt{5}}.$$

3. $\dfrac{a}{x} + \sqrt{\dfrac{a^2 - x^2}{x^2}} = \dfrac{x}{b}$.

Ans. $x' = +\sqrt{2ab - b^2}$ *and* $x'' = -\sqrt{2ab - b^2}$.

4. $\sqrt{\dfrac{x + a}{x}} + 2\sqrt{\dfrac{a}{x + a}} = b^2\sqrt{\dfrac{x}{x + a}}$

Ans. $x' = \dfrac{a}{(b - 1)^2}$ $x'' = \dfrac{a}{(b + 1)^2}$

5. $\dfrac{a - \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}} = b$. *Ans.* $x = \pm\dfrac{2\sqrt{b}}{1 + b}$.

6. $\dfrac{\sqrt{x} + \sqrt{x - a}}{\sqrt{x} - \sqrt{x - a}} = \dfrac{n^2 a}{x - a}$. *Ans.* $x = \dfrac{a(1 \pm n)^2}{1 \pm 2n}$.

7. $\dfrac{\sqrt{a + x}}{\sqrt{x}} + \dfrac{\sqrt{a - x}}{\sqrt{x}} = \sqrt{\dfrac{x}{b}}$. *Ans.* $x = \pm 2\sqrt{ab - b^2}$.

8. $\dfrac{a + x + \sqrt{2ax + x^2}}{a + x} = b$. *Ans.* $x = \dfrac{\pm a(1 \pm \sqrt{2b - b^2})}{\sqrt{2b - b^2}}$

Wednesday

IV. TRINOMIAL EQUATIONS.

132. A Trinomial Equation is one which involves only terms containing two different powers of the unknown quantity, and a known term or terms.

Every trinomial equation can be reduced to the form,

$$x^m + 2px^n = q \quad \ldots\ldots\ldots \quad (1)$$

in which m and n are positive whole members, and p and q known quantities (Art. **120**).

If we make $m = 2$ and $n = 1$, equation (1) becomes,

$$x^2 + 2px = q,$$

a trinomial equation of the second degree.

If we make $m = 2n$, the trinomial equation will take the form

$$x^{2n} + 2px^n = q;$$

and equations of this form may be solved by first regarding x^n as a simple quantity, and afterward extracting the n^{th} root of both members of the equation.

EXAMPLES.

1. Given $x^4 - 25x^2 = -144.$

This is already of the required form, and may be written,

$$(x^2)^2 - 25\,(x^2) = -144.$$

Solving with respect to x^2, regarded as a simple quantity,

$$x^2 = \frac{25}{2} \pm \sqrt{-144 + \frac{625}{4}} = \frac{25 \pm 7}{2}.$$

Extracting the square root of both members,

$$x = \pm\sqrt{\frac{25 \pm 7}{2}}.$$

Hence, the four roots are

$$+4, \qquad -4, \qquad +3, \qquad \text{and} \qquad -3.$$

2. Given $x^2 + \sqrt{x^2 + 11} = 31.$

Transposing and squaring,

$$x^2 + 11 = 961 - 62x^2 + x^4.$$

Reducing,

$$x^4 - 63x^2 = -950.$$

Solving with respect to x^2,

$$x^2 = \frac{63}{2} \pm \frac{13}{2};$$

$$\therefore \quad x = \pm \sqrt{\frac{63 \pm 13}{2}}.$$

Hence, the roots are $+5$, -5, $+\sqrt{38}$, $-\sqrt{38}$.

3. Given $(x^2+5)^2 - 4x^2 = 160.$

Subtracting 20 from both members, and factoring,

$$(x^2+5)^2 - 4(x^2+5) = 140.$$

Solving with respect to $x^2 + 5$,

$$x^2 + 5 = 2 \pm \sqrt{140+4} = 2 \pm 12.$$

Transposing and extracting the square root,

$$x = \pm \sqrt{-3 \pm 12}.$$

4. Given $x + 6\sqrt{x} = 27.$

Solving with respect to $\sqrt{x}$,

$$\sqrt{x} = -3 \pm 6;$$

$$\therefore \quad x = (-3 \pm 6)^2.$$

5. Given $x^2 - 2x + 6\sqrt{x^2 - 2x + 5} = 11.$

Adding 5 to both members,

$$x^2 - 2x + 5 + 6\sqrt{x^2 - 2x + 5} = 16.$$

Solving with respect to $\sqrt{x^2 - 2x + 5}$,

$$\sqrt{x^2 - 2x + 5} = -3 \pm 5.$$

Taking the upper sign and squaring,

$$x^2 - 2x + 5 = 4; \qquad \therefore \qquad x = 1.$$

Taking the lower sign and squaring,

$$x^2 - 2x + 5 = 64; \qquad \therefore \qquad x = 1 \pm 2\sqrt{15}.$$

6. $x^4 - (2bc + 4a^2)x^2 = -b^2c^2$.

Ans. $x = \pm\sqrt{bc + 2a^2 \pm 2a\sqrt{bc + a^2}}$.

7. $2x - 7\sqrt{x} = 99$. *Ans.* $x = 81, \quad x = \frac{121}{4}$.

8. $\frac{a}{b} - bx^4 + \frac{c}{d}x^2 = 0$. *Ans.* $x = \pm\sqrt{\frac{c \pm \sqrt{4ad^2 + c^2}}{2bd}}$.

133. Double Radicals.—The solution of trinomial equations of the fourth degree requires the extraction of the square root of expressions of the form of $a \pm \sqrt{b}$, in which a and b are positive or negative, numerical or algebraic. The expression, $\sqrt{a \pm \sqrt{b}}$ can sometimes be reduced to the form of $a' \pm \sqrt{b'}$, or to the form $\sqrt{a''} \pm \sqrt{b''}$; and when such transformation is possible, it is advantageous to effect it, since, in this case, we have only to extract two simple square roots; whereas, the expression

$$\sqrt{a \pm \sqrt{b}}$$

requires the extraction of the square root of the square root.

To deduce formulas for making the required transformation, let us assume

$$p + q = \sqrt{a + \sqrt{b}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$p - q = \sqrt{a - \sqrt{b}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which p and q are arbitrary quantities.

It is now required to find such values for p and q as will satisfy these equations.

Squaring both members of equations (1) and (2), we have

$$p^2 + 2pq + q^2 = a + \sqrt{b} \quad . \; . \; . \; . \; . \; (3)$$

$$p^2 - 2pq + q^2 = a - \sqrt{b} \quad . \; . \; . \; . \; . \; (4)$$

Adding equations (3) and (4), member to member, we get

$$p^2 + q^2 = a \quad . \; . \; . \; . \; . \; . \; . \; . \; (5)$$

Multiplying (1) and (2), member by member, we have

$$p^2 - q^2 = \sqrt{a^2 - b}.$$

Let us now represent $\sqrt{a^2 - b}$ by c. Substituting in the last equation,

$$p^2 - q^2 = c \quad . \; . \; . \; . \; . \; . \; . \; . \; (6)$$

From (5) and (6) we readily deduce,

$$p = \pm\sqrt{\frac{a+c}{2}} \quad \text{and} \quad q = \pm\sqrt{\frac{a-c}{2}};$$

these values substituted for p and q, in equations (1) and (2), give

$$\sqrt{a + \sqrt{b}} = \pm\sqrt{\frac{a+c}{2}} \pm \sqrt{\frac{a-c}{2}};$$

$$\sqrt{a - \sqrt{b}} = \pm\sqrt{\frac{a+c}{2}} \pm \sqrt{\frac{a-c}{2}}:$$

hence,

$$\sqrt{a + \sqrt{b}} = \pm\left(\sqrt{\frac{a+c}{2}} + \sqrt{\frac{a-c}{2}}\right) \quad . \; . \; (7)$$

and

$$\sqrt{a - \sqrt{b}} = \pm\left(\sqrt{\frac{a+c}{2}} - \sqrt{\frac{a-c}{2}}\right) \quad . \; . \; (8)$$

Now, if $a^2 - b$ is a perfect square, its square root, c, will be a rational quantity, and one of the formulas (7) or (8) will reduce the given expression to the required form. If $a^2 - b$ is not a perfect square, the formulas will not simplify the given expression, for, we shall still have to extract the square root of a square root.

Therefore, this transformation is not used, unless $a^2 - b$ is a perfect square.

EXAMPLES.

1. Reduce $\sqrt{94 + 42\sqrt{5}} = \sqrt{94 + \sqrt{8820}}$ to its simplest form.

We have $$a = 94, \quad b = 8820,$$

$$\therefore \qquad c = \sqrt{a^2 - b} = \sqrt{8836 - 8820} = 4,$$

a rational quantity; formula (7) is therefore applicable to this case, and we have

$$\sqrt{94 + 42\sqrt{5}} = \pm\left(\sqrt{\frac{94 + 4}{2}} + \sqrt{\frac{94 - 4}{2}}\right);$$

or, reducing, $$= \pm(\sqrt{49} + \sqrt{45});$$

hence, $$\sqrt{94 + 42\sqrt{5}} = \pm(7 + 3\sqrt{5}).$$

2. Reduce $\sqrt{np + 2m^2 - 2m\sqrt{np + m^2}}$ to its simplest form.

We have

$$a = np + 2m^2 \quad \text{and} \quad b = 4m^2(np + m^2),$$

$$a^2 - b = n^2p^2 \quad \text{and} \quad c = \sqrt{a^2 - b} = np;$$

and therefore, formula (7) is applicable. It gives,

$$\pm\left(\sqrt{\frac{np + 2m^2 + np}{2}} - \sqrt{\frac{np + 2m^2 - np}{2}}\right),$$

and, reducing, $$\pm(\sqrt{np + m^2} - m).$$

3. Reduce to its simplest form,

$$\sqrt{16 + 30\sqrt{-1}} + \sqrt{16 - 30\sqrt{-1}}.$$

By applying the formulas, we find

$$\sqrt{16 + 30\sqrt{-1}} = 5 + 3\sqrt{-1},$$

and

$$\sqrt{16 - 30\sqrt{-1}} = 5 - 3\sqrt{-1}:$$

hence,

$$\sqrt{16 + 30\sqrt{-1}} + \sqrt{16 - 30\sqrt{-1}} = 10.$$

This example shows that the transformation is applicable to imaginary expressions.

4. Reduce to its simplest form,

$$\sqrt{28 + 10\sqrt{3}}. \qquad \textit{Ans. } 5 + \sqrt{3}.$$

5. Reduce to its simplest form,

$$\sqrt{1 + 4\sqrt{-3}}. \qquad \textit{Ans. } 2 + \sqrt{-3}.$$

6. Reduce to its simplest form,

$$\sqrt{bc + 2b\sqrt{bc - b^2}} - \sqrt{bc - 2b\sqrt{bc - b^2}}.$$

Ans. $\pm 2b$.

7. Reduce to its simplest form,

$$\sqrt{ab + 4c^2 - d^2 - 2\sqrt{4abc^2 - abd^2}}.$$

Ans. $\sqrt{ab} - \sqrt{4c^2 - d^2}$.

V. SOLUTION OF SIMULTANEOUS EQUATIONS.

134. General Case.—It is impossible to solve the general case of two simultaneous equations of the second degree; for, if we combine them, eliminating one of the unknown quantities, the resulting equation will be of the *fourth* degree, and we have not yet learned how to solve the general case of equations of the *fourth* degree. It is always possible to solve two simultaneous equations when one is of the second degree and the other of the first degree; and there are many instances in which we can solve *particular* cases of simultaneous equations when both are of the second degree.

135. First.—To solve two simultaneous equations when one is of the first and the other of the second degree, *we find the value of one of the unknown quantities in terms of the other, from the first equation, and substitute this in the second equation; there will result an equation of the second degree, which can be solved by known methods. The value of the other quantity can then be found by substitution.*

EXAMPLES.

1. Given
$$\begin{cases} 2x - y = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1) \\ x^2 + 2y^2 = 22 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2) \end{cases}$$

We find from (1),

$$x = \frac{1+y}{2}; \quad \therefore \quad x^2 = \frac{1 + 2y + y^2}{4}.$$

Substituting in (2),

$$\frac{1 + 2y + y^2}{4} + 2y^2 = 22.$$

Solving with respect to y,

$$y' = 3 \qquad \text{and} \qquad y'' = -\frac{29}{9}.$$

Finding the corresponding values of x from (1), we have,

$$x' = 2 \qquad \text{and} \qquad x'' = -\frac{10}{9}.$$

2. $x + 2y = 7$
$x^2 + 3xy + y^2 = 31$ *Ans.* $\begin{cases} x' = 3, \; x'' = 25. \\ y' = 2, \; y'' = -9. \end{cases}$

3. $2x + y = 27$
$3xy = 210$ *Ans.* $\begin{cases} x' = 3\frac{1}{2}, \; x'' = 10. \\ y' = 20, \; y'' = 7. \end{cases}$

4. $2x - 3y - 1 = 0$
$2x^2 + xy - 5y^2 = 20$ *Ans.* $\begin{cases} x' = 5, \; x'' = -9\frac{1}{4}. \\ y' = 3, \; y'' = -6\frac{1}{2}. \end{cases}$

5. $\dfrac{10x + y}{xy} = 3$
$y - x = 2$ *Ans.* $\begin{cases} x' = 2, \; x'' = -\frac{1}{3}. \\ y' = 4, \; y'' = +\frac{5}{3}. \end{cases}$

136. Second.—We can always solve two equations of the second degree containing two unknown quantities when they are both homogeneous with respect to these quantities.

For, *we can substitute for one of the quantities an auxiliary unknown quantity multiplied into the second unknown quantity, and by combining the two resulting equations, can find an equation of the second degree, from which the value of the auxiliary unknown quantity may be determined, and thence the values of the required quantities can be found.*

EXAMPLES.

6. Given $\begin{cases} x^2 + xy - y^2 = 5 \; . \; . \; . \; . \; . \; . \; . \; . \quad (1) \\ 3x^2 - 2xy - 2y^2 = 6 \; . \; . \; . \; . \; . \; . \; . \; . \quad (2) \end{cases}$

Substituting px for y, p being unknown, we have,

$$x^2 + px^2 - p^2x^2 = 5 \; . \; . \; . \; . \; . \; . \; . \quad (3)$$
$$3x^2 - 2px^2 - 2p^2x^2 = 6 \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

Finding the values of x^2 in terms of p, from equations (3) and (4), and placing them equal to each other, we deduce

$$\frac{5}{1+p-p^2}=\frac{6}{3-2p-2p^2}.$$

Reducing,

$$p^2+4p=\frac{9}{4};$$

whence,

$$p=\frac{1}{2} \quad \text{and} \quad p=-\frac{9}{2}.$$

Considering the positive value of p, we have, by substituting it in equation (3),

$$x^2\left(1+\frac{1}{2}-\frac{1}{4}\right)=5,$$

or,

$$x^2=4;$$

whence,

$$x'=2 \qquad \text{and} \qquad x''=-2;$$

and since $y=px$, we have $y'=1$ and $y''=-1$.

7. $x^2-xy=6$ $\quad$ $x^2+xy=66$ $\quad$ *Ans.* $\begin{cases} x'=6, & x''=-6. \\ y'=5, & y''=-5. \end{cases}$

8. $x^2-xy=48$ $\quad$ $xy-y^2=12$ $\quad$ *Ans.* $\begin{cases} x=\pm 8, \\ y=\pm 2. \end{cases}$

9. $x^2+4xy+4y^2=256$ $\quad$ $3y^2-x^2=39$ $\quad$ *Ans.* $\begin{cases} x'=\pm 6, & x''=\pm 102. \\ y'=\pm 5, & y''=\mp 59. \end{cases}$

10. $6(x^2+y^2)=13xy$ $\quad$ $x^2-y^2=20$ $\quad$ *Ans.* $\begin{cases} x=\pm 6. \\ y=\pm 4. \end{cases}$

137. Third.—There are also many cases which admit of solution, but for which no fixed rule can be given. A few such examples are given, with such explanations as may suggest the general methods of proceeding.

EXAMPLES.

11. Given, $$\left.\begin{aligned} \frac{xy}{\sqrt{\frac{x}{y}}} &= 48, \\ \frac{xy}{\sqrt{x}} &= 24, \end{aligned}\right\} \text{ to find } x \text{ and } y.$$

Dividing the first by the second, member by member, we have,

$$\frac{\sqrt{x}}{\sqrt{\frac{x}{y}}} = 2, \quad \text{or} \quad \sqrt{y} = 2; \quad \text{whence,} \quad y = 4.$$

Substituting in the second equation, we get

$$\sqrt{x} = 6 \quad \text{and} \quad x = 36.$$

12. Given $$\left.\begin{aligned} x + \sqrt{xy} + y &= 19, \\ x^2 + xy + y^2 &= 133, \end{aligned}\right\} \text{ to find } x \text{ and } y.$$

Dividing the second by the first, member by member, we have,

$$x - \sqrt{xy} + y = 7.$$

But,

$$x + \sqrt{xy} + y = 19:$$

adding these, member to member, and dividing by 2, we find,

$$x + y = 13,$$

which substituted in the first equation, gives,

$$\sqrt{xy} = 6, \quad \text{or,} \quad xy = 36, \quad \text{and,} \quad x = \frac{36}{y}.$$

Substituting this for x, in the preceding equation, we get,

$$\frac{36}{y} + y = 13;$$

hence,

$$y^2 - 13y = -36;$$

whence,

$$y = \frac{13}{2} \pm \sqrt{-36 + \frac{169}{4}} = \frac{13}{2} \pm \frac{5}{2}:$$

Finally,

$$y' = 9, \qquad \text{or} \qquad y'' = 4;$$

but

$$x + y = 13;$$

hence,

$$x' = 4, \qquad \text{or} \qquad x'' = 9.$$

13. Find the values of x and y in the equations,

$$x^2 + 3x + y = 73 - 2xy \quad . \; . \; . \; . \; . \; . \quad (1)$$
$$y^2 + 3y + x = 44 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

By transposition, the first equation becomes,

$$x^2 + 2xy + 3x + y = 73;$$

to which, if the second be added, member to member, there results,

$$x^2 + 2xy + y^2 + 4x + 4y = (x + y)^2 + 4(x + y) = 117.$$

If, now, in the equation

$$(x + y)^2 + 4(x + y) = 117,$$

we regard $x + y$ as a single unknown quantity, we have

$$x + y = -2 \pm \sqrt{117 + 4};$$
$$\therefore \quad x + y = -2 + 11 = 9,$$

and,

$$x + y = -2 - 11 = -13;$$

hence, $\quad x = 9 - y, \quad$ and $\quad x = -13 - y.$

Substituting these values of x in equation (2), we have

$$y^2 + 2y = 35,$$
$$y^2 + 2y = 57.$$

The first gives

$$y' = 5 \qquad \text{and} \qquad y'' = -7,$$

and the second

$$y' = -1 + \sqrt{58} \qquad \text{and} \qquad y'' = -1 - \sqrt{58}.$$

The corresponding values of x are

$$x' = 4, \qquad x'' = 16;$$

$$x' = -12 - \sqrt{58} \qquad \text{and} \qquad x'' = -12 + \sqrt{58}.$$

14. Find the values of x and y in the equations,

$$x^2y^2 + xy^2 + xy = 600 - (y + 2)\,x^2y^3 \quad . \; . \; . \quad (1)$$
$$x + y^2 = 14 - y \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

From the first equation, we have

$$x^2y^2 + (y^2 + 2y)\,x^2y^2 + xy^2 + xy = 600,$$
$$\text{or} \qquad x^2y^2(1 + y^2 + 2y) + xy\,(1 + y) = 600,$$
$$\therefore \qquad x^2y^2\,(1 + y)^2 + xy\,(1 + y) = 600.$$

Solving with respect to $xy\,(1 + y)$, we have

$$xy\,(1 + y) = -\tfrac{1}{2} \pm \sqrt{600 + \tfrac{1}{4}} = -\tfrac{1}{2} \pm \sqrt{\frac{2401}{4}}.$$

If we discuss the roots which belong to the positive value of the radical, we have

$$xy\,(1 + y) = -\frac{1}{2} + \frac{49}{2} = 24;$$

and hence,

$$x = \frac{24}{y + y^2} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

Substituting this value for x in equation (2), we have

$$(y^2 + y)^2 - 14\,(y^2 + y) = -24;$$
$$y^2 + y = 12 \quad \text{and} \quad y^2 + y = 2.$$

From equation (1), we have

$$y = -\frac{1}{2} \pm \frac{7}{2};$$

the corresponding value of x from equation (3), is

$$x = 2.$$

From the second equation, we have

$$y' = 1 \qquad \text{and} \qquad y'' = -2;$$

which give

$$x = 12.$$

15. Given $x^2y + xy^2 = 6$ and $x^3y^2 + x^2y^3 = 12$, to find the values of x and y.

$$Ans. \begin{cases} x = 2 \text{ and } 1, \\ y = 1 \text{ and } 2. \end{cases}$$

16. Given $\left\{\begin{matrix} x^2 + x + y = 18 - y^2 \\ xy = 6 \end{matrix}\right\}$ to find the values of x and y.

$$Ans. \begin{cases} x = 3, \text{ or } 2; \text{ and } -3 \pm \sqrt{3}, \\ y = 2, \text{ or } 3; \text{ and } -3 \mp \sqrt{3}. \end{cases}$$

VI. PRACTICAL PROBLEMS.

1. Find two numbers such, that the sum of the products of the first by a, and the second by b, is equal to $2s$; and the product of one by the other is equal to p.

Let x and y denote the two numbers. Then from the conditions of the problem, we have

$$ax + by = 2s,$$
$$xy = p.$$

From the first, we have

$$y = \frac{2s - ax}{b}.$$

Substituting in the second, we have, after reduction,

$$ax^2 - 2sx = -bp;$$

$$\therefore \qquad x = \frac{s}{a} \pm \frac{1}{a}\sqrt{s^2 - abp}; \qquad \text{and}$$

$$y = \frac{s}{b} \mp \frac{1}{b}\sqrt{s^2 - abp}$$

If, in this problem, we suppose $a = b = 1$, the values of x and y become

$$x = s \pm \sqrt{s^2 - p} \qquad \text{and} \qquad y = s \mp \sqrt{s^2 - p};$$

hence we see that, under this supposition, the two values of x are equal to those of y, taken in an inverse order; which shows, that if

$s + \sqrt{s^2 - p}$ represent the value of x, $s - \sqrt{s^2 - p}$

will represent the corresponding value of y, and the reverse. This is as it should be; for under this hypothesis the first two equations become

$$x + y = 2s, \qquad \text{and}$$
$$xy = p.$$

The question is then, *to find two numbers whose sum is s and whose product is p.*

2. Find *four* numbers such, that the sum of the *first* and *fourth* shall be equal to $2s$, the sum of the *second* and *third* to $2s'$, the sum of their squares equal to $4c^2$, and the product of the first and fourth equal to the product of the second and third.

Let u, x, y, and z, denote the numbers, respectively. Then, from the conditions of the problem, we have

$$u + z = 2s \quad \ldots\ldots\ldots \quad (1)$$
$$x + y = 2s' \quad \ldots\ldots\ldots \quad (2)$$
$$u^2 + x^2 + y^2 + z^2 = 4c^2 \quad \ldots\ldots\ldots \quad (3)$$
$$uz = xy \quad \ldots\ldots\ldots \quad (4)$$

Let p be the product of the 1st and 4th, or 2d and 3d; we then have

$$\left.\begin{array}{l} u + z = 2s, \\ uz = p, \end{array}\right\} \text{ which give } \left\{\begin{array}{l} u = s + \sqrt{s^2 - p}, \\ z = s - \sqrt{s^2 - p}; \end{array}\right.$$

and $x + y = 2s'$, which give $x = s' + \sqrt{s'^2 - p}$,

$xy = p$, $y = s' - \sqrt{s'^2 - p}$.

Substituting these values of u, x, y, z, in equation (3), it becomes

$$(s + \sqrt{s^2 - p})^2 + (s - \sqrt{s^2 - p})^2 + (s' + \sqrt{s'^2 - p})^2 + (s' - \sqrt{s'^2 - p})^2 = 4c^2;$$

developing and reducing, we have

$$4s^2 + 4s'^2 - 4p = 4c^2; \quad \text{hence,} \quad p = s^2 + s'^2 - c^2.$$

Substituting this value for p, in the expressions for u, x, y, z, we find

$$\begin{cases} u = s + \sqrt{c^2 - s'^2}, \\ z = s - \sqrt{c^2 - s'^2}, \end{cases} \qquad \begin{cases} x = s' + \sqrt{c^2 - s^2}, \\ y = s' - \sqrt{c^2 - s^2}. \end{cases}$$

These values satisfy the equation (4); for

$$uz = (s + \sqrt{c^2 - s'^2})\,(s - \sqrt{c^2 - s'^2}) = s^2 - c^2 + s'^2,$$
$$xy = (s' + \sqrt{c^2 - s^2})\,(s' - \sqrt{c^2 - s^2}) = s'^2 - c^2 + s^2.$$

This problem shows the advantage of introducing an *auxiliary unknown quantity.* There are problems which lead to equations of a higher degree than the second, but which may be resolved by the methods of solving equations of the second degree, as in the following:

3. Given the sum of two numbers equal to a, and the sum of their cubes equal to c, to find the numbers

By the conditions $\begin{cases} x + y = a \\ x^3 + y^3 = c. \end{cases}$

Putting $x = s + z$ and $y = s - z$, we have $a = 2s$, and

$$\begin{cases} x^3 = s^3 + 3s^2z + 3sz^2 + z^3 \\ y^3 = s^3 - 3s^2z + 3sz^2 - z^3: \end{cases}$$

hence, by addition,

$$x^3 + y^3 = 2s^3 + 6sz^2 = c;$$

hence,

$$z^2 = \frac{c - 2s^3}{6s}, \quad \text{and} \quad z = \pm\sqrt{\frac{c - 2s^3}{6s}},$$

or,

$$x = s \pm \sqrt{\frac{c - 2s^3}{6s}}, \quad \text{and} \quad y = s \mp \sqrt{\frac{c - 2s^3}{6s}};$$

substituting for s its value,

$$x = \frac{a}{2} \pm \sqrt{\left(\frac{c - \frac{1}{4}a^3}{3a}\right)} = \frac{a}{2} \pm \sqrt{\frac{4c - a^3}{12a}},$$

and

$$y = \frac{a}{2} \mp \sqrt{\left(\frac{c - \frac{1}{4}a^3}{3a}\right)} = \frac{a}{2} \mp \sqrt{\frac{4c - a^3}{12a}}.$$

4. The sum of the squares of two numbers is equal to a, and the difference of their squares to b: what are the numbers?

Ans. $\sqrt{\frac{a + b}{2}}$, $\sqrt{\frac{a - b}{2}}$.

5. Find three numbers which, when multiplied two and two, and each product divided by the third number, give the quotients, a, b, c.

Ans. $\sqrt{ab}$, $\sqrt{ac}$, $\sqrt{bc}$.

6. The sum of two numbers is 8, and the sum of their cubes is 152: what are the numbers? *Ans.* 3 and 5.

7. Find two numbers, whose difference added to the difference of their squares is 150, and whose sum added to the sum of their squares, is 330. *Ans.* 9 and 15.

8. There are two numbers whose difference is 15, and the half of whose product is equal to the cube of the lesser number: what are the numbers? *Ans.* 3 and 18.

9. What two numbers are those whose sum multiplied by

the greater, is equal to 77; and whose difference, multiplied by the less, is equal to 12?

Ans. 4 and 7, or $\frac{3}{2}\sqrt{2}$ and $\frac{11}{2}\sqrt{2}$.

10. Divide 100 into two parts, such that the sum of their square roots is 14. *Ans.* 64 and 36.

Friday

11. It is required to divide 24 into two such parts, that their product shall be equal to 35 times their difference.

Ans. 10 and 14.

12. Find two numbers whose product is 255, and the sum of whose squares is 514. *Ans.* 15 and 17.

Friday

13. There is a number expressed by two digits, which, when divided by the sum of the digits, gives a quotient greater by 2 than the first digit; but if the digits be inverted, and the resulting number be divided by a number greater by 1 than the sum of the digits, the quotient will exceed the former quotient by 2: what is the number?

Ans. 24.

14. A regiment, in garrison, consisting of a certain number of companies, receives orders to send 216 men on duty, each company to furnish an equal number. Before the order is executed, three of the companies are sent on other service, and it is then found that each company that remains has to send 12 men additional, to make up the complement, 216. How many companies were in the regiment, and what number of men did each of the remaining companies send?

Ans. 9 companies: each that remained sent 36 men.

15. Find three numbers such, that their sum shall be 14, the sum of their squares 84, and the product of the first and third equal to the square of the second.

Ans. 2, 4, and 8.

16. It is required to find a number, expressed by three digits, such, that the sum of the squares of the digits shall

be 104; the square of the middle digit to exceed twice the product of the other two by 4; and if 594 be subtracted from the number, the remainder will be expressed by the same digits reversed. *Ans.* 862.

17. A person has three kinds of goods which together cost \$$230\frac{5}{24}$. A pound of each article costs as many $\frac{1}{24}$ dollars as there are pounds in that article: he has one-third more of the second than of the first, and $3\frac{1}{2}$ times as much of the third as of the second: How many pounds has he of each?
Ans. 15 of the 1st, 20 of the 2d, 70 of the 3d.

18. Two merchants sold the same kind of stuff: the second sold 3 yards more than the first, and together, they received 35 dollars. The first said to the second, "I would have received 24 dollars for your stuff." The other replied, "And I would have received $12\frac{1}{2}$ dollars for yours." How many yards did each sell?

Ans. 1st merchant 15 or 5
2d " 18 or 8.

19. A widow possessed 13,000 dollars, which she divided into two parts, and placed both at interest, so that the incomes from them were equal. If she had put out the first part at the same rate as the second, she would have drawn for this part 360 dollars interest; and if she had put out the second at the same rate as the first, she would have drawn for it 490 dollars interest. What were the two rates of interest? *Ans.* 7 and 6 per cent.

the greater, is equal to 77; and whose difference, multiplied by the less, is equal to 12?

Ans. 4 and 7, or $\frac{3}{2}\sqrt{2}$ and $\frac{11}{2}\sqrt{2}$.

10. Divide 100 into two parts, such that the sum of their square roots is 14. *Ans.* 64 and 36.

Friday

11. It is required to divide 24 into two such parts, that their product shall be equal to 35 times their difference.

Ans. 10 and 14.

12. Find two numbers whose product is 255, and the sum of whose squares is 514. *Ans.* 15 and 17.

Friday

13. There is a number expressed by two digits, which, when divided by the sum of the digits, gives a quotient greater by 2 than the first digit; but if the digits be inverted, and the resulting number be divided by a number greater by 1 than the sum of the digits, the quotient will exceed the former quotient by 2: what is the number?

Ans. 24.

14. A regiment, in garrison, consisting of a certain number of companies, receives orders to send 216 men on duty, each company to furnish an equal number. Before the order is executed, three of the companies are sent on other service, and it is then found that each company that remains has to send 12 men additional, to make up the complement, 216. How many companies were in the regiment, and what number of men did each of the remaining companies send?

Ans. 9 companies: each that remained sent 36 men.

15. Find three numbers such, that their sum shall be 14, the sum of their squares 84, and the product of the first and third equal to the square of the second.

Ans. 2, 4, and 8.

16. It is required to find a number, expressed by three digits, such, that the sum of the squares of the digits shall

be 104; the square of the middle digit to exceed twice the product of the other two by 4; and if 594 be subtracted from the number, the remainder will be expressed by the same digits reversed. *Ans.* 862.

17. A person has three kinds of goods which together cost \$$230\frac{5}{24}$. A pound of each article costs as many $\frac{1}{24}$ dollars as there are pounds in that article: he has one-third more of the second than of the first, and $3\frac{1}{2}$ times as much of the third as of the second: How many pounds has he of each?

Ans. 15 of the 1st, 20 of the 2d, 70 of the 3d.

18. Two merchants sold the same kind of stuff: the second sold 3 yards more than the first, and together, they received 35 dollars. The first said to the second, "I would have received 24 dollars for your stuff." The other replied, "And I would have received $12\frac{1}{2}$ dollars for yours." How many yards did each sell?

Ans. 1st merchant 15 or 5
2d " 18 or 8.

19. A widow possessed 13,000 dollars, which she divided into two parts, and placed both at interest, so that the incomes from them were equal. If she had put out the first part at the same rate as the second, she would have drawn for this part 360 dollars interest; and if she had put out the second at the same rate as the first, she would have drawn for it 490 dollars interest. What were the two rates of interest? *Ans.* 7 and 6 per cent.

CHAPTER VII.

FORMATION OF POWERS AND EXTRACTION OF ROOTS OF ALL DEGREES.

I. Powers of Monomials.

138. A **Power** of a quantity, is the product that results from taking that quantity any number of times as a factor.

To deduce a rule for forming any power of a monomial, let it be required to find the third power of $7a^2b^3c$. From the definition of a power and the rule for multiplication, we have

$$(7a^2b^3c)^3 = 7a^2b^3c \times 7a^2b^3c \times 7a^2b^3c = 343a^6b^9c^3.$$

In the same manner any monomial may be raised to any power. Hence, the following

RULE.

Raise the coefficient to the required power, and multiply the exponent of each letter by the exponent of the required power.

If the sign of a quantity is +, the sign of any power of that quantity will be +. If the sign of a quantity is —, the signs of the *even* powers of that quantity are +, and the signs of the *odd* powers are —.

EXAMPLES.

Raise the following monomials to the powers indicated:

1. $(2a^3)^3$. *Ans.* $8a^9$.

2. $(-2a^2b)^5$. *Ans.* $-32a^{10}b^5$.

3. $(3x^2yz^2)^3$. *Ans.* $27x^6y^3z^6$.

4. $(-2a^3b^2c)^4$. *Ans.* $16a^{12}b^8c^4$.

5. $(-3a^{-2}bc^{-3})^3$. *Ans.* $-27a^{-6}b^3c^{-9}$.

6. $(-8a^2b^{-2}c^3)^4$. *Ans.* $4096a^8b^{-8}c^{12}$.

139. Powers of Fractions.—Let it be required to find the third power of the monomial fraction $\frac{2a^2bc}{3xy^2}$. From the definition of a power and the rule for multiplication of fractions, we have

$$\left(\frac{2a^2bc^3}{3xy^2}\right)^3 = \frac{2a^2bc^3}{3xy^2} \times \frac{2a^2bc^3}{3xy^2} \times \frac{2a^2bc^3}{3xy^2} = \frac{8a^6b^3c^9}{27x^3y^6};$$

and in like manner we may raise any fraction to any power. Hence, the following

RULE.

Raise the numerator to the required power for a new numerator, and the denominator to the required power for a new denominator.

EXAMPLES.

Raise the following fractions to the powers indicated:

1. $\left(-\frac{2a^2b^3}{3}\right)^4$. *Ans.* $\frac{16a^8b^{12}}{81}$.

2. $\left(-\frac{3a^4c^2}{4b^3}\right)^3$. *Ans.* $-\frac{27a^{12}c^6}{64b^9}$.

3. $\left(-\frac{x^2yz^3}{2}\right)^4$. *Ans.* $\frac{x^8y^4z^{12}}{16}$.

4. $\left(\frac{2ax^{-2}}{3yz}\right)^3$. *Ans.* $\frac{8a^3x^{-6}}{27y^3z^3}$.

5. $\left(-\frac{3ay^4}{2b^2x}\right)^3$. *Ans.* $-\frac{27a^3y^{12}}{8b^6x^3}$.

140. Powers of Binomials.—A binomial may be raised to any power by the process of continued multiplication; but when the required power is higher than the second, the operation may be considerably abridged by using the *Binomial Formula.*

II. THE BINOMIAL FORMULA.

141. The Binomial Formula is a formula by means of which a binomial may be raised to any power without going through with the operation of continued multiplication.

To deduce a formula for finding the value of $(x+y)^m$, when m is any positive whole number, let us assume the following results deduced by actual multiplication:

$$
\begin{aligned}
(x+y)^1 &= x + y,\\
(x+y)^2 &= x^2 + 2xy + y^2,\\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3,\\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.
\end{aligned}
$$

If we examine these powers, we find that they are all formed according to the following laws:

1°. **The Law of Exponents.**—*The exponent of the leading letter in the first term is equal to the exponent of the power; and the exponent of that letter goes on diminishing by 1 in each term toward the right, to the last term, where it is 0; the exponent of the following letter is 0 in the first term, and the exponent of that letter goes on increasing by 1 in each term toward the right, to the last term, where it is equal to the exponent of the power.*

2°. **The Law of Coefficients.**—*The coefficient of the first term is 1; the coefficient of any term after the first is found by multiplying the coefficient of the preceding term by the exponent of the leading letter in that term, and dividing the product by the number of terms that precede the required term.*

Let us assume that these laws hold true for the formation of a power whose exponent is m, m being any positive whole number. The application of these laws gives,

$$(x+y)^m = x^m + mx^{m-1}y + m.\frac{m-1}{2}x^{m-2}y^2 + m.\frac{m-1}{2}.\frac{m-2}{3}x^{m-3}y^3 + \&c. + y^m.$$

Let both members of this equation be multiplied by $(x+y)$; the first member will become $(x+y)^{m+1}$; to find what the second member will become, let us perform the multiplication.

OPERATION.

$$\begin{array}{l}
x^m + mx^{m-1}y + m.\frac{m-1}{2}x^{m-2}y^2 + \&c. + y^m \\
x + y \\
\hline
x^{m+1} + mx^m y + m.\frac{m-1}{2}x^{m-1}y^2 + \&c. + xy^m \\
\qquad\qquad x^m y + \qquad\qquad mx^{m-1}y^2 + \&c. + mxy^m + y^{m+1} \\
\hline
\end{array}$$

$$\begin{array}{l|l|l|l}
x^{m+1} + m & x^m y + m.\frac{m-1}{2} & x^{m-1}y^2 + \&c. + m & xy^m + y^{m+1} \\
\quad\;\; +1 & \qquad\;\; + m & \qquad\qquad\qquad\quad +1 &
\end{array}$$

But

$$m.\frac{m-1}{2} + m = m\left(\frac{m-1}{2} + 1\right) = \frac{(m+1)m}{1.2}.$$

$$m.\frac{m-1}{2}.\frac{m-2}{3} + m.\frac{m-1}{2} = m.\frac{m-1}{2}\left(\frac{m-2}{3} + 1\right)$$

$$= \frac{(m+1)m(m-1)}{1.2.3}.$$

$$m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}+m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}$$
$$=\frac{(m+1)\,m(m-1)\,(m-2)}{1.2.3.4}.$$

The law of coefficients is evident. Substituting in the product just found, we have,

$$(x+y)^{m+1}=x^{m+1}+(m+1)\,x^{m}y+\frac{(m+1)m}{1.2}\,x^{m-1}y^{2}$$
$$+\frac{(m+1)\,m(m-1)}{1.2.3}\,x^{m-2}y^{3}+\&c.+y^{m+1}.$$

Examining this result, we see that it conforms to the assumed laws of formation. Hence, we have shown that, *if the assumed laws hold good for the formation of any power, they must also hold good for the formation of the next higher power.* But we have proved that they hold good for the formation of the *fourth power;* hence, they must hold good for the formation of the *fifth* power. But if they hold good for the formation of the *fifth* power, they must, from the principle demonstrated, hold good for the formation of the *sixth* power; and if for the *sixth,* then for the *seventh;* and so on, by successive deduction, it may be shown that they hold good for *any power* whose exponent is a positive whole number.

But we have already supposed m to represent any positive whole number. Hence, we have proved the following formula:

$$(x+y)^{m}=x^{m}+mx^{m-1}y+m\cdot\frac{m-1}{2}x^{m-2}y^{2}+\&c.+mxy^{m-1}+y^{m}\quad(1)$$

which is the *binomial formula.*

This formula, which has only been proved true when m is a positive whole number, will be shown hereafter to hold good when m is either *positive* or *negative, entire* or *fractional.*

If we change the places of x and y, in formula (1), and then apply the laws of formation, the order of the terms will be reversed, and we shall have,

$$(y+x)^m = y^m + my^{m-1}x + m\cdot\frac{m-1}{2}y^{m-2}x^2 + \&c. + myx^{m-1} + x,^m \quad (2)$$

Comparing the second members of (1) and (2), we see that corresponding coefficients are equal; which shows that *the coefficients of the second member of the binomial formula, taken at equal distances from the extremes, are equal.* Hence, in forming any power of a binomial, it is only necessary to find the coefficients to the middle term of the development; the remaining ones will be equal to these taken in a reverse order.

From the laws of formation, we see that there are $m + 1$ terms in the development of the m^{th} power of a binomial. Hence, when m is odd, we must find $\frac{m+1}{2}$ coefficients; when m is even, we must find $\frac{m+2}{2}$ coefficients.

To apply the formula to find any power of a binomial, we raise the first term to its successive powers as high as the m^{th}, and substitute them for the corresponding powers of x in the formula; we then raise the second term to its successive powers as high as the m^{th}, and substitute them in the formula for the corresponding powers of y, substituting also for m the exponent of the power.

EXAMPLES.

1. Find the 6th power of $x + a$.

Here we replace y by a and m by 6. Hence,

$$(x+a)^6 = x^6 + 6ax^5 + 15a^2x^4 + 20a^3x^3 + 15a^4x^2 + 6a^5x + a^6.$$

The same result would have been found by applying *the law of exponents* and *the law of coefficients.* The simplest manner of applying these laws will now be shown:

$$\begin{array}{lllllllll}
\text{Powers of } x & \ldots & x^6 & +x^5 & +x^4 & +x^3 & +x^2 & +x & +1 \\
\text{Powers of } a & \ldots & 1 & +a & +a^2 & +a^3 & +a^4 & +a^5 & +a^6 \\
\text{Coefficients} & \ldots & 1 & +6 & +15 & +20 & +15 & +6 & +1 \\
\hline
\end{array}$$

$$(x+a)^6 = x^6 + 6ax^5 + 15a^2x^4 + 20a^3x^3 + 15a^4x^2 + 6a^5x + a^6$$

In the *first* line are written the descending powers of x, and in the *second* line the ascending powers of a, both in accordance with *the law of exponents.* In the third line are written the coefficients of the different terms deduced from *the law of coefficients.* The final result is found by multiplying together the factors standing in the different vertical columns.

The coefficients in the third line are the coefficients of the *formula,* and may be deduced by giving to m its proper value. In this particular case $m = 6$; hence, the coefficient of the second term is 6; that of the third term is $\frac{6 \times 5}{2} = 15$; that of the fourth term is $\frac{6 \times 5 \times 4}{2 \times 3} = 20$; and so on.

The same method of proceeding may be applied when the terms of the binomial contain coefficients and exponents.

2. Find the fourth power of $3a^2c - 2bd$.

$$\begin{array}{lllllll}
\text{Powers of } 3a^2c & \ldots & 81a^8c^4 & +\ 27a^6c^3 & +\ 9a^4c^2 & +\ 3a^2c & +\ 1 \\
\text{Powers } -2bd & \ldots & 1 & -\ 2bd & +\ 4b^2d^2 & -\ 8b^3d^3 & +16b^4d^4 \\
\text{Coefficients} & \ldots & 1 & +\ 4 & +\ 6 & +\ 4 & +\ 1 \\
\hline
\end{array}$$

$$(3a^2c - 2bd)^4 = 81a^8c^4 - 216a^6bc^3d + 216a^4b^2c^2d^2 - 96a^2b^3cd^3 + 16b^4d^4$$

Care must be taken to distinguish between the coefficients of the formula, written in the third line, and the coefficients of the development, in the fourth line.

3. Find the fourth power of $1 + \frac{x}{2}$.

$$\begin{array}{lllllll}
\text{Powers of } 1 & \ldots & 1 & +\ 1 & +\ 1 & +\ 1 & +\ 1 \\
\text{Powers of } \frac{x}{2} & \ldots & 1 & +\ \frac{x}{2} & +\ \frac{x^2}{4} & +\ \frac{x^3}{8} & +\ \frac{x^4}{16} \\
\text{Coefficients} & \ldots & 1 & +\ 4 & +\ 6 & +\ 4 & +\ 1 \\
\hline
\end{array}$$

$$\left(1 + \frac{x}{2}\right)^4 = 1 + 2x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{x^4}{16}.$$

4. Find the third power of $2a^2 - 3x^3$.

$$\begin{array}{llllllll}
\text{Powers of } 2a^2 \ . \ . & 8a^6 & + & 4a^4 & + & 2a^2 & + & 1 \\
\text{Powers of } -3x^3 \ . & 1 & - & 3x^3 & + & 9x^6 & - & 27x^9 \\
\text{Coefficients} \ . \ . \ . & 1 & + & 3 & + & 3 & + & 1 \\
\hline
(2a^2 - 3x^3)^3 = & 8a^6 & - & 36a^4x^3 & + & 54a^2x^6 & - & 27x^9.
\end{array}$$

In writing the first line it will be most convenient to begin with the 0 power, that is, at the right hand; the second line will be most conveniently written from left to right.

5. Find the 5th power of $2x + 3y$.

Ans. $32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$.

6. Find the 4th power of $a^2 - x^2$.

Ans. $a^8 - 4a^6x^2 + 6a^4x^4 - 4a^2x^6 + x^8$.

7. Find the 5th power of $1 + \frac{x}{2}$.

Ans. $1 + \frac{5x}{2} + \frac{5x^2}{2} + \frac{5x^3}{4} + \frac{5x^4}{16} + \frac{x^5}{32}$.

8. Find the 5th power of $1 + 4x$.

Ans. $1 + 20x + 160x^2 + 640x^3 + 1280x^4 + 1024x^5$.

Powers of Polynomials.

142. A **Power** of any polynomial may be found by means of the binomial formula, as in the following examples:

1. Let it be required to find the third power of

$$a + b + c.$$

First, put $$b + c = d;$$

then $(a + b + c)^3 = (a + d)^3 = a^3 + 3a^2d + 3ad^2 + d^3$.

Substituting for d, its value, we have

$$\begin{array}{rl}
(a + b + c)^3 = a^3 + 3a^2b + 3ab^2 + b^3 & \\
3a^2c + 3b^2c + 6abc & \\
+ 3ac^2 + 3bc^2 & \\
+ c^3. &
\end{array}$$

2. This development is composed of, *the cubes of the thre terms, plus three times the square of each term by the firs powers of the two others, plus six times the product of al three terms.*

To apply the preceding formula to the development of the cube of a trinomial, in which the terms are affected with coefficients and exponents, *designate each term by a single letter, and perform the operations indicated; then replace the letters introduced, by their values.*

From this rule, we find that

$$(2a^2 - 4ab + 3b^2)^3 = 8a^6 - 48a^5b + 132a^4b^2 - 208a^3b^3 \\ + 198a^2b^4 - 108ab^5 + 27b^6.$$

The fourth, fifth, etc., powers of any polynomial can be developed in a similar manner.

The results in all these cases may also be found by linking the terms so as to form a binomial, and then proceeding as already explained. Thus, let it be required to find the square of the polynomial $a + b + c + d$, or $(a + b) + (c + d)$.

$$\begin{array}{llllll}
\text{Powers of } (a+b)\ldots & (a^2+2ab+b^2) & & (a+b) & + & 1 \\
\text{Powers of } (c+d)\ldots & 1 & + & (c+d) & + & (c^2+2cd+d^2) \\
\text{Coefficients}\ldots\ldots\ldots & 1 & + & 2 & + & 1 \\
\hline
\end{array}$$

$$(a+b+c+d)^2 = a^2+2ab+b^2+2ac+2bc+2ad+2bd+c^2+2cd+d^2.$$

Arranging and factoring, we have

$$(a+b+c+d)^2 = a^2+2a(b+c+d)+b^2+2b(c+d)+c^2+2cd+d^2.$$

In like manner it may be shown that the square of any polynomial is equal to

The square of its first term, plus *twice the product of the first term into the sum of all that follow it,* plus *the square of the second term,* plus *twice the product of the second term into the sum of all that follow it; and so on, to the square of the last term.*

III. EXTRACTION OF ROOTS OF NUMBERS.

143. Cube Root of Numbers.—The *cube root of a number* is one of its three equal factors.

A Perfect Cube is a number whose cube root can be exactly found; all other numbers are *imperfect cubes.*

The first ten whole numbers are,

1, 2, 3, 4, 5, 6, 7, 8, 9, 10;

and their cubes,

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000.

Conversely, the numbers in the first line are the cube roots of the corresponding numbers in the second.

If we wish to find the cube root of any number, less than 1000, we look for the number in the second line, and if it is found there, the corresponding number in the first line will be its cube root. If the number is not found there, it will fall between two numbers in the second line, and its cube root will fall between the corresponding numbers in the first line. In this case the cube root cannot be expressed in exact parts of 1; hence, the given number is an imperfect cube.

If the given number is greater than 1000, its cube root will be greater than 10; that is, it will contain a certain number of tens and a certain number of units.

Let us designate such a whole number by N, and denote its tens by a, and its units by b; we shall have,

$N = a + b$; whence, $N^3 = a^3 + 3a^2b + 3ab^2 + b^3$; that is,

The cube of a number is equal to the cube of the tens, plus three times the product of the square of the tens by the units, plus three times the product of the tens by the square of the units, plus the cube of the units.

Thus $(47)^3=(40)^3+3\times(40)^2\times7+3\times40\times(7)^2+(7)^3=10382$

Let us now reverse the operation, and find the cube roo of 103823.

$$\begin{array}{r|l} 10\dot{3}\,82\dot{3} & \underline{47} \\ 64 & 8 \\ 4^2\times3=48 \,\overline{)\,398\,23} & \end{array}$$

$$\begin{array}{rr} 48 & 47 \\ \underline{48} & \underline{47} \\ 384 & 329 \\ \underline{192} & \underline{188} \\ 2304 & 2209 \\ \underline{48} & \underline{47} \\ 18432 & 15463 \\ \underline{9216} & \underline{8836} \\ 110592 & 103823 \end{array}$$

Since the number is greater than 1000, its root will contain tens and units. We will first find the number of tens in the root. Now the cube of tens, giving at least thousands, we point off three places of figures on the right, and the cube of the number of tens will be found in the number 103, to the left of this period.

The cube root of the greatest cube contained in 103 being 4, this is the number of tens in the required root.

Having found the number of tens, subtract its cube, 64, from 103, and there remains 39, to which bring down the part 823, and we have 39823, which contains *three times the product of the square of the tens by the units, plus three times the product of the tens by the square of the units, plus the cube of the units.*

As the square of the tens gives at least hundreds, it follows that the product of three times the square of the tens by the units, must be found in the part 398, to the left of 23, which is separated from it by a dash. Therefore, dividing 398 by 48, which is three times the square of the tens, the quotient 8 will be the units of the root, or something greater, since 398 is composed of three times the square of the tens y the units, and generally contains numbers coming from he two other parts.

We may ascertain whether the figure 8 is too great, by orming from the 4 tens and 8 units, the three parts which nter into 39823; but it is much easier to cube 48, as has een done in the above table. Now, the cube of 48 is 110592, hich is greater than 103823; therefore, 8 is too great. By ubstituting 7 for 8 and cubing 47, we obtain 103823; hence, he proposed number is a perfect cube, and 47 is its cube oot.

By a course of reasoning entirely analogous to that purued in treating of the extraction of the square root, we ay show that, when the given number is expressed by more han six figures, we must point off the number into periods f three figures each, commencing at the right. Hence, for he extraction of the cube root of a whole number, we have the following

RULE.

I. Separate the given number into periods of three figures each, beginning at the right; the left hand period will often contain less than three places of figures:

II. Seek the greatest perfect cube in the first period, on the left, and set its root on the right, after the manner of a quotient in division. Subtract the cube of this number from the first period, and to the remainder bring down the first figure of the next period, and call this number a dividend:

III. Take three times the square of the root just found for a divisor, and see how often it is contained in the dividend, and place the quotient for a second figure of the root. Then cube the number thus found, and if its cube be greater than the first two periods of the given number, diminish the last figure by 1; but if it be less, subtract it from the first two periods, *and to the remainder bring down the first figure of the next period, for a new dividend:*

IV. Take three times the square of the root found for a new divisor, and seek how often it is contained in the new dividend; the quotient will be the third figure of the root. Cube the number thus found, and subtract the result from the first three periods of the given number, and proceed in a similar way until all the periods are operated upon.

If there is no remainder, the number is a *perfect cube*, and the root is exact: if there is a remainder, the number is an *imperfect cube*, and the root is exact to within less than one. If any trial figure of the root is too great, diminish it.

EXAMPLES.

1. $\sqrt[3]{48228544}$ *Ans.* 364.
2. $\sqrt[3]{27054036008}$ *Ans.* 3002.
3. $\sqrt[3]{483249}$ *Ans.* 78, with a remainder 8697.
4. $\sqrt[3]{91632508641}$ *Ans.* 4508, with a rem. 20644129.
5. $\sqrt[3]{32977340218432}$ *Ans.* 32068.

144. The n^{th} Root of a Number.—The n^{th} root of a number is one of its n equal factors. When the n^{th} root of a number can be found exactly, the number is a *perfect n^{th} power*; all other numbers are *imperfect n^{th} powers.*

Let a table of n^{th} powers be formed, similar to that employed in the last article; then if the number contain less than n figures, and is a perfect n^{th} power, it will be found in the second line, and its n^{th} root, which will consist of a single figure, will be found in the first line. If the number is not a perfect n^{th} power, it will fall between two perfect n^{th} powers in the second line, and its n^{th} root will fall between the corresponding numbers in the first line.

If the given number contain more than n figures, its root will consist of a certain number of tens and a certain number of units. If we designate the tens of the root by a, the units by b, and the number by N, we shall have, by the binomial formula,

$$N = (a + b)^n = a^n + na^{n-1}b + n\frac{n-1}{2}a^{n-2}b^2 + \&c.;$$

that is, the proposed number is equal to *the* n^{th} *power of the tens, plus* n *times the product of the* $(n-1)^{th}$ *power of the tens by the units,* plus other parts not necessary to consider.

Now, as the n^{th} power of the tens cannot be less than 1 followed by n ciphers, the last n figures on the right cannot make a part of it. They must then be pointed off, and the n^{th} root of the greatest n^{th} power in the number on the left will be the tens of the required root.

Subtract the n^{th} power of the number of tens from the number on the left, and to the remainder bring down one figure of the next period on the right. If we consider the number thus found as a dividend, and take n times the $(n-1)^{th}$ power of the tens, as a divisor, the quotient will evidently be the number of units, or a greater number.

If the part on the left should contain more than n figures, the n figures on the right of it must be separated from the rest, and the root of the greatest n^{th} power contained in the part on the left extracted, and so on. Hence the following

RULE.

I. Separate the number N into periods of n figures each, beginning at the right hand; find the n^{th} *root of the greatest perfect* n^{th} *power in the left-hand period, and it will be the first figure of the root:*

II. Subtract this n^{th} *power from the left-hand period, and bring down the first figure of the next period, and call this the dividend:*

III. Form the $n-1$ *power of the first figure of the root, multiply it by* n*, and see how many times* *the product is contained in the dividend: the quotient will be the second figure of the root, or something greater:*

IV. Raise the number thus formed to the n^{th} power, then subtract this result from the two left-hand periods, and to the new remainder bring down the first figure of the next period: then divide the number thus formed by n times the $n-1$ power of the root already found, and continue this operation until all the periods are brought down and operated upon.

EXAMPLES.

1. What is the fourth root of 531441?

$$\begin{array}{rl} & 5\dot{3}\,144\dot{1} \mid 27 \\ 2^4 = & \underline{16} \\ 4 \times 2^3 = 32 \mid & 371 \\ (27)^4 = & 531441 \end{array}$$

We first point off, from the right hand, the period of four figures, and then find the greatest fourth root contained in 53, the first period to the left, which is 2. We next subtract the 4th power of 2, which is 16, from 53, and to the remainder, 37, we bring down the first figure of the next period. We then divide 371 by 4 times the cube of 2, which gives 11 for a quotient: but this we know is too large. By trying the numbers 9 and 8, we find them also too large: then trying 7, we find the exact root to be 27.

145. Index a Multiple.—When the index of the root to be extracted is a multiple of two or more numbers, as 4, 6, . . . etc., *the root can be obtained by extracting roots of more simple degrees, successively.* To explain this, we will remark that,

$$(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3 = a^{3+3+3+3} = a^{3\times4} = a^{12};$$

and, in general, from the definition of an exponent

$$(a^m)^n = a^m \times a^m \times a^m \times a^m \ldots = a^{m\times n}:$$

hence, *the n^{th} power of the m^{th} power of a number is equal to the mn^{th} power of this number.*

The converse of this is also true; for let

$$\sqrt[n]{\sqrt[m]{a}} = b;$$

raising both members to the n^{th} power, we have

$$\sqrt[m]{a} = b^n;$$

raising both members of this equation to the m^{th} power, we have

$$a = b^{mn}.$$

Extracting the mn^{th} root of both members, we have

$$\sqrt[mn]{a} = b;$$

and since things equal to the same thing are equal to each other, we have

$$\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}.$$

Therefore, *the n^{th} root of the m^{th} root of any number, is equal to the mn^{th} root of that number.*

In a similar manner, it may be proved that

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$$

By this method we find that

1. $\sqrt[4]{256} = \sqrt{\sqrt{256}} = \sqrt{16} = 4.$

2. $\sqrt[6]{2985984} = \sqrt[3]{\sqrt{2985984}} = \sqrt[3]{1728} = 12.$

3. $\sqrt[6]{1771561} = \sqrt[3]{\sqrt{1771561}} = 11.$

4. $\sqrt[8]{1679616} = \sqrt[4]{1296} = \sqrt{\sqrt{1296}} = 6.$

Although the successive roots may be extracted in any order, it is better to extract the roots of the lower degree first, for then the extraction of the roots of the higher degrees, which is a more complicated operation, is applied to numbers containing fewer figures than the proposed number.

146. Roots of Fractions.—Since the n^{th} power of a fraction is formed by raising both terms of the fraction to the n^{th} power, we can evidently find the n^{th} root of a fraction by extracting the n^{th} root of both terms.

If both terms are not perfect n^{th} powers, the exact n^{th} root cannot be found, but we may find its approximate root to within less than the *fractional unit,* as follows:

Let $\frac{a}{b}$ represent the given fraction. If we multiply both terms by b^{n-1} it becomes,

$$\frac{a}{b}=\frac{ab^{n-1}}{b^n}.$$

Let r denote the n^{th} root of the greatest perfect n^{th} power in ab^{n-1}; then $\frac{ab^{n-1}}{b^n}$ will be comprised between $\frac{r^n}{b^n}$ and $\frac{(r+1)^n}{b^n}$; and consequently, $\frac{r}{b}$ will be the n^{th} root of $\frac{a}{b}$ to within less than $\frac{1}{b}$; hence,

Multiply the numerator by the $(n-1)^{th}$ power of the denominator, and extract the n^{th} root of the product: Divide this by the denominator of the fraction, and the quotient will be the approximate root.

147. Approximation.—When it is required to extract the n^{th} root of a number which is not a *perfect power,* the method already explained will give only the entire part of the root, or the root to within less than 1. The part to be added, in order to complete the root, cannot be obtained exactly, but we can approximate to it as near as we please.

Let it be required to extract the n^{th} root of a whole number, denoted by a, to within less than a fraction $\frac{1}{p}$; that is, so that the error shall be less than $\frac{1}{p}$.

We can write

$$a = \frac{ap^n}{p^n}.$$

If we denote by r, the root of ap^n to within less than 1, the number $\frac{a \times p^n}{p^n} = a$, will be comprehended between $\frac{r^n}{p^n}$ and $\frac{(r+1)^n}{p^n}$; therefore, the $\sqrt[n]{a}$ will be comprised between $\frac{r}{p}$ and $\frac{r+1}{p}$; and consequently, their difference $\frac{1}{p}$ will be greater than the difference between $\frac{r}{p}$ and the true root. Hence, $\frac{r}{p}$ is the required root to within less than $\frac{1}{p}$; hence,

To extract the n^{th} root of a whole number to within less than $\frac{1}{p}$, multiply the number by p^n; extract the n^{th} root of the product to within less than 1, and divide the result by p.

EXAMPLES.

1. Required the cube root of 15 to within less than $\frac{1}{12}$. We have

$$15 \times 12^3 = 15 \times 1728 = 25920.$$

The cube root of 25920, to within less than 1, is 29; hence, the required root is,

$$\frac{29}{12} = 2\frac{5}{12}.$$

2. Extract the cube root of 47, to within less than $\frac{1}{20}$. We have

$$47 \times 20^3 = 47 \times 8000 = 376000.$$

The cube root of 376000, to within less than 1, is 72; hence,

$$\sqrt[3]{47} = \frac{72}{20} = 3\frac{12}{20}, \quad \text{to within less than } \frac{1}{20}.$$

3. Find the value of $\sqrt[3]{25}$, to within less than .001.

Multiply 25 by the cube of 1000, or 1000000000, which gives 25000000000. The cube root of this number is 2920; hence,

$$\sqrt[3]{25} = 2.920 \text{ to within less than } .001.$$

Since other examples may be treated in a similar manner, we have, for extracting the cube root of a number to within a given decimal, the following

RULE

Annex three times as many ciphers to the number, as there are decimal places in the required root; extract the cube root of the number thus formed to within less than 1, and point off from the right of this root the required number of decimal places.

148. Decimals.—We will now explain the method of *extracting the cube root of a decimal fraction.*

Suppose it is required to extract the cube root of 3.1415.

Since the denominator, 10000, of this fraction, is not a perfect cube, make it one, by multiplying it by 100; *this is equivalent to annexing two ciphers to the proposed decimal,* which then becomes, 3.141500. Extract the cube root of 3141500, that is, of the number considered independent of the decimal point to within less than 1; this gives 146. Then dividing by 100, or by $\sqrt[3]{1000000}$, and we find,

$$\sqrt[3]{3.1415} = 1.46, \quad \text{to within less than } 0.01.$$

Since similar examples may be treated in a similar manner, we have the following

RULE.

Annex ciphers till the whole number of decimal places is three times the number required in the root. Then extract the root as in whole numbers, and point off the required number of decimal places.

Vulgar Fraction.—To extract the cube root of a vulgar fraction to within less than a given decimal, the most simple method is,

To reduce the proposed fraction to a decimal, continuing the division until the number of decimal places is equal to three times the number required in the root.

The question is then reduced to extracting the cube root of a decimal fraction.

Suppose it is required to find the sixth root of 23, to within less than 0.01.

Applying the rule of Art. **147** to this example, we multiply 23 by $(100)^6$, or annex *twelve* ciphers to 23; then extract the sixth root of the number thus formed to within less than 1, and divide this root by 100, or point off two decimal places: we thus find,

$$\sqrt[6]{23} = 1.68, \text{ to within less than } 0.01.$$

EXAMPLES.

1. Find the $\sqrt[3]{473}$ to within less than $\frac{1}{20}$. *Ans.* $7\frac{3}{4}$.
2. Find the $\sqrt[3]{79}$ to within less than .0001. *Ans.* 4.2908.
3. Find the $\sqrt[6]{13}$ to within less than .01. *Ans.* 1.53.
4. Find the $\sqrt[3]{3.00415}$ to within less than .0001. *Ans.* 1.4429.
5. Find the $\sqrt[3]{0.00101}$ to within less than .01. *Ans.* 0.10.
6. Find the $\sqrt[3]{\frac{14}{25}}$ to within less than .001. *Ans.* 0.824.

IV. EXTRACTION OF ROOTS OF ALGEBRAIC QUANTITIES.

149. To find a rule for extracting any root of a monomial, we have simply to reverse the rule of Article **138**, for finding any power of a monomial. This gives for the extraction of the n^{th} root of a monomial the following

RULE.

Extract the n^{th} root of the coefficient, and divide the exponent of each letter by n.

Hence, that a monomial may be a perfect n^{th} power:

1°. Its coefficient must be a perfect n^{th} power; and

2°. The exponent of each letter must be divisible by n.

EXAMPLES.

Extract the indicated roots of the following monomials:

1. $\sqrt[3]{64a^9b^3c^6}$. *Ans.* $+4a^3bc^2$.

2. $\sqrt[4]{16a^8b^{12}c^4}$. *Ans.* $\pm 2a^2b^3c$.

It may be shown, as in Art. **111**, that any even power of a positive quantity is +, and also that any even power of a negative quantity is +; hence, every even root must have the double sign ±. Every odd power of a quantity has the same sign as the quantity; hence, every odd root of a quantity must have the same sign as the quantity.

3. $\sqrt[3]{-8a^3}$. *Ans.* $-2a$.

4. $\sqrt[5]{-32a^{10}b^5}$. *Ans.* $-2a^2b$.

5. $\sqrt[4]{81a^4b^{12}}$. *Ans.* $\pm 3ab^3$.

6. $\sqrt[3]{8a^6b^3c^{12}}$. *Ans.* $+2a^2bc^4$.

7. $\sqrt[4]{81a^4b^8c^{16}}$. *Ans.* $\pm 3ab^2c^4$.

8. $\sqrt[5]{-32a^5c^{10}d^{15}}$. *Ans.* $-2ac^2d^3$.

9. $\sqrt[3]{-125a^9b^6c^3}$. *Ans.* $-5a^3b^2c$.

Any even root of a negative quantity is imaginary; for it is impossible to find a quantity which, on being raised to an even power, will give a negative result.

10. $\sqrt[4]{-16a^4}$. *Ans.* $2a\sqrt[4]{-1}$.

150. Roots of Polynomials.—To deduce a rule for extracting the n^{th} root of a polynomial, n being any positive whole number, let us first consider the law of formation of the n^{th} power of a polynomial. For this purpose take the polynomial

$$r + r' + r'' + \&c.,$$

which suppose arranged with respect to some letter. Denoting the sum of all the terms after the first by s, and applying the binomial formula, we have

$$\begin{aligned}(r + s)^n &= r^n + nr^{n-1}s + \&c.,\\ &= r^n + nr^{n-1}(r' + r'' + \&c.) + \&c.\end{aligned}$$

If now we reverse the process, and extract the n^{th} root of the second member, it is obvious that we shall find the first term of the root by extracting the n^{th} root of the first term of the polynomial. If we subtract r^n from the polynomial, and denote the remainder by R, we have

$$R = nr^{n-1}r' + nr^{n-1}r'' + \&c.,$$

which remainder will evidently be arranged with reference to the leading letter of the polynomial; therefore, the first term will contain a higher power of that letter than either of the succeeding terms, and cannot be reduced with any of them. Hence, if we divide the first term of the first remainder, by

n times the $(n-1)^{th}$ power of the first term of the root, the quotient will be the second term of the root.

In like manner, if we subtract the n^{th} power of the sum of the first two terms of the root from the given polynomial, and divide the first term of the remainder by n times the $(n-1)^{th}$ power of the first term of the root, we shall have the third term of the root. If we continue the operation, the first term of each new remainder, divided by n times the $(n-1)^{th}$ power of the first term of the root, will give a new term of the root.

It may be remarked, since the first term of the first remainder is the same as the second term of the given polynomial, that we can find the second term of the root, by dividing the second term of the given polynomial by n times the $(n-1)^{th}$ power of the first term.

Hence, for the extraction of the n^{th} root of a polynomial, we have the following

RULE.

I. Arrange the given polynomial with reference to one of its letters, and extract the n^{th} root of the first term; this will be the first term of the root:

II. Divide the second term by n times the $(n-1)^{th}$ power of the first term; the quotient will be the second term:

III. Subtract the n^{th} power of the sum of the two terms already found from the given polynomial, and divide the first term of the remainder by n times the $(n-1)^{th}$ power of the first term; the quotient will be the third term:

IV. Continue this operation till a remainder is found equal to 0, or, till one is found whose first term is not divisible by n times the $(n-1)^{th}$ power of the first term of the root: in the former case the root is exact, and the given polynomial a perfect n^{th} power; in the latter case, the polynomial is an imperfect n^{th} power.

EXAMPLES.

1. Extract the cube root of

$$x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1.$$

OPERATION.

$$\begin{array}{r|l}
x^6-6x^5+15x^4-20x^3+15x^2-6x+1 & \underline{x^2-2x+1} \\
(x^2-2x)^3=\underline{x^6-6x^5+12x^4-\ 8x^3} \qquad\qquad & \underline{3x^4,} \text{ Divisor.} \\
3x^4-\ \&c. \qquad\qquad\qquad & \\
(x^2-2x+1)^3=\underline{x^6-6x^5+15x^4-20x^3+15x^2-6x+1} & \\
0. \qquad\qquad\qquad\quad &
\end{array}$$

In this example, we first extract the cube root of x^6, which gives x^2, for the first term of the root. Squaring x^2, and multiplying by 3, we obtain the divisor $3x^4$: this is contained in the second term $-2x$ times. Then cubing the part of the root found, and subtracting, we find that the first term of the remainder $3x^4$, contains the divisor once. Cubing the whole root found, we find the cube equal to the given polynomial. Hence, $x^2 - 2x + 1$, is the exact cube root.

2. Find the cube root of

$$x^6 + 6x^5 - 40x^3 + 96x - 64.$$

Ans. $x^2 + 2x - 4$.

3. Find the cube root of

$$8x^6 - 12x^5 + 30x^4 - 25x^3 + 30x^2 - 12x + 8.$$

Ans. $2x^2 - x + 2$.

4. Find the 4th root of

$$16a^4 - 96a^3x + 216a^2x^2 - 216ax^3 + 81x^4.$$

Ans. $2a - 3x$.

We first extract the 4th root of $16a^4$, which is $2a$. We then raise $2a$ to the third power, and multiply by 4, the index of the root; this gives the divisor $32a^3$. This divisor is contained in the second term, $-3x$ times, which is the second term of the root. Raising the root found to the 4th power, we find the result equal to the given polynomial.

5. Find the fourth root of

$$81x^8 - 216x^7 + 336x^5 - 56x^4 - 224x^3 + 64x + 16.$$

Ans. $3x^2 - 2x - 2$.

V. TRANSFORMATION OF RADICALS.

151. The following principles are used in transforming radicals of any degree:

Let a and b denote any two quantities, and p the product of their n^{th} roots. Then,

$$\sqrt[n]{a} \times \sqrt[n]{b} = p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Raising both members of this equation to the n^{th} power, we find

$$(\sqrt[n]{a})^n \times (\sqrt[n]{b})^n = p^n, \quad \text{or} \quad ab = p^n;$$

whence, by extracting the n^{th} root of both members,

$$\sqrt[n]{ab} = p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since the second members of equations (1) and (2) are the same, their first members are equal; therefore,

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}; \text{ hence,}$$

1°. *The product of the n^{th} roots of two quantities is equal to the n^{th} root of the product of the quantities, and the reverse.*

Denoting the quotient of the given radicals by q, we have

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Raising both members to the n^{th} power, we have

$$\frac{(\sqrt[n]{a})^n}{(\sqrt[n]{b})^n} = q^n, \quad \text{or} \quad \frac{a}{b} = q^n.$$

Extracting the n^{th} root of both members, we have

$$\sqrt[n]{\frac{a}{b}} = q \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

The second members of (1) and (2) are the same; hence, their first members are equal, giving

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}; \text{ hence,}$$

2°. *The quotient of the n^{th} roots of two quantities is equal to the n^{th} root of their quotient, and the reverse.*

Let a denote any quantity, and assume

$$\sqrt[m]{\sqrt[n]{a}} = r \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (1)$$

Raising both members to the m^{th} power, we have

$$\sqrt[n]{a} = r^m \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2)$$

Raising both members of (2) to the n^{th} power, we have

$$a = r^{mn} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (3)$$

Extracting the mn^{th} root of both members of (3), we have

$$\sqrt[mn]{a} = r \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (4)$$

But the second members of (1) and (4) are the same; their first members are therefore equal; hence,

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}; \text{ hence,}$$

3°. *The mth root of the nth root of any quantity is equal to the mnth root of that quantity, and the reverse.*

152. Radicals of any Degree.—A radical of the nth degree is in its *simplest* form when there is no factor under the radical sign that is a perfect nth power.

Take the radical, $\sqrt[3]{54a^4b^3c^2}$. This may be written,

$$\sqrt[3]{54a^4b^3c^2} = \sqrt[3]{27a^3b^3 \times 2ac^2}.$$

Hence, from principle 1° (Art. **151**), we have

$$\sqrt[3]{54a^4b^3c^2} = \sqrt[3]{27a^3b^3} \times \sqrt[3]{2ac^2} = 3ab\sqrt[3]{2ac^2}.$$

Since we may treat any radical in a similar manner, we have, for the simplification of a radical of the nth degree, the following

RULE.

Resolve the quantity under the radical sign into two factors, one of which is the greatest perfect nth power that enters it, as a factor: extract the nth root of this factor, and write the root without the radical sign, under which, leave the other factor.

Conversely, *a coefficient may be introduced under the radical sign, by simply raising it to the nth power, and writing it as a factor under the radical sign.*

Thus, $3ab\sqrt[3]{2ac^2} = \sqrt[3]{27a^3b^3} \times \sqrt[3]{2ac^2} = \sqrt[3]{54a^4b^3c^2}$.

EXAMPLES.

Simplify the following radicals:

1. $\sqrt[4]{48a^5b^2c^4}$. *Ans.* $2ac\sqrt[4]{3ab^2}$.
2. $\sqrt[3]{112a^4b^6}$. *Ans.* $2ab^2\sqrt[3]{14a}$.
3. $2\sqrt[3]{54x^{-4}y^{-6}}$. *Ans.* $6x^{-1}y^{-2}\sqrt[3]{2x^{-1}}$.
4. $\sqrt[4]{512a^{4m}y^{-5}}$. *Ans.* $4a^my^{-1}\sqrt[4]{2y^{-1}}$.

5. $3\sqrt[3]{128x^5y^{-5}}$. *Ans.* $4xy^{-1}\sqrt[3]{2x^2y^{-2}}$.

R 6. $\sqrt[3]{-24x^{-3}y^5}$. *Ans.* $-2x^{-1}y\sqrt[3]{3y^2}$.

Place the coefficients under the radical sign in the following expressions:

7. $3\sqrt[3]{2ax}$. *Ans.* $\sqrt[3]{54ax}$.

8. $-4\sqrt[4]{ab^2}$. *Ans.* $\sqrt[4]{256ab^2}$.

9. $2ab\sqrt[4]{2a^{-3}y^{-5}}$. *Ans.* $\sqrt[4]{32ab^4y^{-5}}$.

R 10. $-3x^{-2}\sqrt[3]{-2y^2}$. *Ans.* $\sqrt[3]{54x^{-6}y^2}$.

153. A radical may sometimes be simplified by reducing its index, in accordance with principle 3° (Art. **151**). Take, for example, the radical $\sqrt[6]{4a^2}$. From the principle referred to, we have

$$\sqrt[6]{4a^2} = \sqrt[3]{\sqrt{4a^2}}.$$

As the quantity under the last radical sign is a perfect square, its root can be extracted; hence,

$$\sqrt[6]{4a^2} = \sqrt[3]{2a}.$$

In like manner,

$$\sqrt[4]{36a^2b^2} = \sqrt{\sqrt{36a^2b^2}} = \sqrt{6ab}.$$

In general,

$$\sqrt[mn]{a^n} = \sqrt[m]{\sqrt[n]{a^n}} = \sqrt[m]{a};$$

that is, when the index of a radical is a multiple of any number n, and the quantity under the radical sign is an exact n^{th} power, *we can divide its index by n, provided we extract the n^{th} root of the quantity under the sign.*

Conversely, *The index of a radical may be multiplied by any number, provided we raise the quantity under the sign to a power of which this number is the exponent.*

For, since a is the same thing as $\sqrt[n]{a^n}$, we have

$$\sqrt[m]{a} = \sqrt[m]{\sqrt[n]{a^n}} = \sqrt[mn]{a^n}.$$

EXAMPLES.

Simplify the following radicals:

1. $\sqrt[4]{36a^2b^2}$. *Ans.* $\sqrt{6ab}$.
2. $\sqrt[6]{16a^4x^2y^{-4}}$. *Ans.* $\sqrt[3]{4a^2xy^{-2}}$.
3. $\sqrt[6]{27m^{12}n^6p^3}$. *Ans.* $\sqrt{3m^4n^2p}$.
4. $\sqrt[6]{-8a^{3m}y^6}$. *Ans.* $-\sqrt{2a^m x^2}$.
5. $\sqrt[9]{8x^3y^{-3n}}$. *Ans.* $\sqrt[3]{2xy^{-n}}$.

154. Common Index of Radicals.—The principles of the last article enable us to reduce radicals having different indices to equivalent ones having a common index.

Take, for examples, the radicals,

$$\sqrt[3]{c}, \quad \sqrt[4]{2a}, \quad \text{and} \quad \sqrt[6]{a+b}.$$

The least common multiple of 3, 4, and 6, is 12. We can multiply the index of each radical by such a number as will make it 12, provided we raise the quantity under the sign to a corresponding power. Thus, we have

$$\sqrt[3]{c} = \sqrt[12]{c^4}, \quad \sqrt[4]{2a} = \sqrt[12]{8a^3}, \quad \text{and} \quad \sqrt[6]{a+b} = \sqrt[12]{(a+b)^2}.$$

In like manner, other groups of radicals may be reduced to a common index. Hence, the following

RULE.

I. Find the least common multiple of the indices, for a common index:

II. Then raise the quantity under each radical sign to a power whose exponent is the quotient of the common index, divided by the index of its radical.

EXAMPLES.

Reduce the following radicals to a common index:

1. $\sqrt[4]{a}$, $\sqrt[6]{5b}$, and $\sqrt[8]{a^2+b^2}$.
Ans. $\sqrt[24]{a^6}$, $\sqrt[24]{(5b)^4}$, and $\sqrt[24]{(a^2+b^2)^3}$.

2. $\sqrt[4]{3}$, $\sqrt[6]{7}$, and $\sqrt{2}$.
Ans. $\sqrt[12]{27}$, $\sqrt[12]{49}$, and $\sqrt[12]{64}$.

3. $2\sqrt[3]{3x}$, $6\sqrt[6]{4x}$, and $\sqrt[4]{a}$.
Ans. $2\sqrt[12]{81x^4}$, $6\sqrt[12]{16x^2}$, and $\sqrt[12]{a^3}$.

4. $\sqrt[3]{a^n}$, $\sqrt[4]{b^m}$, and $\sqrt[6]{c}$.
Ans. $\sqrt[12]{a^{4n}}$, $\sqrt[12]{b^{3m}}$, and $\sqrt[12]{c^2}$.

5. $\sqrt[m]{a}$, $\sqrt[n]{b}$, and $\sqrt[p]{c}$.
Ans. $\sqrt[mnp]{a^{np}}$, $\sqrt[mnp]{b^{mp}}$, and $\sqrt[mnp]{c^{mn}}$.

6. $\sqrt{\dfrac{1}{a-b}}$ and $\sqrt[3]{\dfrac{1}{x-y}}$.
Ans. $\sqrt[6]{\dfrac{1}{(a-b)^3}}$ and $\sqrt[6]{\dfrac{1}{(x-y)^2}}$.

155. Addition and Subtraction of Radicals.—Radicals can only be added and subtracted when they are *similar*, that is, when they have the same index and the same quantity under the radical sign. When they are similar, the radical part may be regarded as a *unit*, and their addition or subtraction may be effected by the following

RULE.

Add or subtract the coefficients, and to the result annex the common radical part.

If two radicals are not similar, they may often be made so by simplification. Hence, we first reduce the radicals to be operated on, to their simplest form, and if not similar, we indicate the addition or subtraction.

EXAMPLES.

Add the following radicals:

1. $\sqrt{48ab^2}$ and $b\sqrt{75a}$. *Ans.* $9b\sqrt{3a}$.

2. $3\sqrt[6]{4a^2}$ and $2\sqrt[3]{2a}$. *Ans.* $5\sqrt[3]{2a}$.

3. $2\sqrt{45}$ and 3 $\sqrt[2]{5}$. ✓ *Ans.* $9\sqrt{5}$.

Reduce the following expressions:

4. $3a\sqrt[4]{b} - 2c\sqrt[4]{b}$. *Ans.* $(3a - 2c)\sqrt[4]{b}$.

5. $3\sqrt[6]{4a^2} - 2\sqrt[3]{2a}$. *Ans.* $\sqrt[3]{2a}$.

6. $\sqrt{243} + \sqrt{27} + \sqrt{48}$. *Ans.* $16\sqrt{3}$.

7. $2\sqrt{8a^3} - 7a\sqrt{18a} + 5\sqrt{72a^3} - \sqrt{50ab^2}$.
Ans. $(13a - 5b)\sqrt{2a}$.

8. $12\sqrt[3]{\frac{1}{4}} + 3\sqrt[3]{\frac{1}{32}}$. *Ans.* $\frac{27}{4}\sqrt[3]{2}$.

9. $\sqrt[3]{8a^3b + 16a^4} - \sqrt[3]{b^4 + 2ab^3}$.
Ans. $(2a - b)\sqrt[3]{2a + b}$.

10. $3\sqrt[6]{4a^2} + 2\sqrt[3]{2a}$. *Ans.* $5\sqrt[3]{2a}$.

11. $\sqrt{45c^3} - \sqrt{80c^3} + \sqrt{5a^2c}$. *Ans.* $(a - c)\sqrt{5c}$.

12. $2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} + \sqrt{\frac{3}{5}}$. *Ans.* $\frac{28}{15}\sqrt{15}$.

156. Multiplication of Radicals.—We have shown that all radicals may be reduced to equivalent ones having a common index; we therefore suppose this transformation made.

Let $a\sqrt[n]{b}$ and $c\sqrt[n]{d}$ denote any two radicals of the same degree. Their product is denoted thus,

$$a\sqrt[n]{b} \times c\sqrt[n]{d};$$

or, since the order of the factors may be changed without affecting the value of the product, we may write it (Art. **151**),

$$ac \times \sqrt[n]{b} \times \sqrt[n]{d} = ac\sqrt[n]{bd};$$

hence, for the multiplication of radicals of any degree, we have the following

RULE.

I. Reduce the radicals to a common index:

II. Multiply the coefficients together for a new coefficient, and the quantities under the radical sign for a new quantity under the radical sign, leaving the index unchanged.

EXAMPLES.

Perform the following indicated multiplications:

1. $2a\sqrt[3]{\frac{a^2+b^2}{c}} \times -3a\sqrt[3]{\frac{(a^2+b^2)^2}{d}}$.

Ans. $-\frac{6a^2(a^2+b^2)}{\sqrt[3]{cd}}$.

2. $3a\sqrt[4]{8a^3} \times 2b\sqrt[4]{4a^2c}$. *Ans.* $12a^2b\sqrt[4]{2c}$.

3. $\frac{3}{4}\sqrt[3]{\frac{1}{3}} \times \frac{1}{4}\sqrt[3]{\frac{1}{7}}$. *Ans.* $\frac{3}{16}\sqrt[3]{\frac{1}{21}}$

4. $3a\sqrt[6]{b} \times 5b\sqrt[8]{2c}$. *Ans.* $15ab\sqrt[24]{8b^4c^3}$.

5. Multiply $\sqrt{2} \times \sqrt[3]{3}$ by $\sqrt[4]{\frac{1}{2}} \times \sqrt[3]{\frac{1}{3}}$.

Ans. $\sqrt[12]{8}$.

6. Multiply $2\sqrt{15}$ by $3\sqrt[3]{10}$.

Ans. $6\sqrt[6]{337500}$.

7. Multiply $4\sqrt[5]{\frac{2}{3}}$ by $2\sqrt{\frac{3}{4}}$. *Ans.* $8\sqrt[10]{\frac{27}{256}}$.

8. Multiply $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$, together.

Ans. $\sqrt[12]{648000}$.

9. Multiply $\sqrt[7]{\frac{4}{3}}$, $\sqrt[3]{\frac{1}{2}}$, and $\sqrt[14]{6}$, together.

Ans. $\sqrt[42]{\frac{2}{27}}$.

10. Multiply $\left(4\sqrt{\frac{7}{3}}+5\sqrt{\frac{1}{2}}\right)$ by $\left(\sqrt{\frac{7}{3}}+2\sqrt{\frac{1}{2}}\right)$.

Ans. $\frac{43}{3}+\frac{13}{6}\sqrt{42}$.

11. Multiply $-\sqrt{\frac{9m^2}{a^2+b^2}}$ by $-\frac{a^2+b^2}{3m}$.

Ans. $\sqrt{a^2+b^2}$.

12. Multiply $\sqrt{x}+\sqrt{y}$ by $\sqrt{x}-\sqrt{y}$. Ans. $x-y$.

157. Division of Radicals.—We will suppose, as in the last article, that the radicals have been reduced to equivalent ones having a common index.

Let $a\sqrt[n]{b}$ and $c\sqrt[n]{d}$ represent any two radicals of the n^{th} degree. The quotient of the first by the second may be written,

$$\frac{a\sqrt[n]{b}}{c\sqrt[n]{d}}=\frac{a}{c}\times\frac{\sqrt[n]{b}}{\sqrt[n]{d}}.$$

But from Art. **151**, we have

$$\frac{\sqrt[n]{b}}{\sqrt[n]{d}}=\sqrt[n]{\frac{b}{d}}.$$

Consequently, we have

$$\frac{a\sqrt[n]{b}}{c\sqrt[n]{d}}=\frac{a}{c}\times\sqrt[n]{\frac{b}{d}}.$$

Hence, to divide one radical by another, we have the following

RULE.

I. Reduce the radicals to equivalent ones having a common index:

II. Divide the coefficient of the dividend by that of the divisor for a new coefficient, and the quantity under the radical sign in the dividend by that in the divisor, for a new quantity under the radical sign, leaving the index unchanged.

EXAMPLES.

1. Find the quotient of $c\sqrt[3]{a^2b^2+b^4}$ by $d\sqrt[3]{\frac{a^2-b^2}{8b}}$.

$$\frac{c}{d}\times\frac{\sqrt[3]{a^2b^2+b^4}}{\sqrt[3]{\frac{a^2-b^2}{8b}}}=\frac{c}{d}\sqrt[3]{\frac{8b\,(a^2b^2+b^4)}{a^2-b^2}}=\frac{2cb}{d}\sqrt[3]{\frac{a^2+b^2}{a^2-b^2}}.$$

2. Divide $2\sqrt{3}\times\sqrt[3]{4}$ by $\frac{1}{2}\sqrt[4]{2}\times\sqrt[3]{3}$.

Ans. $4\sqrt[12]{288}$.

3. Divide $\frac{1}{2}\sqrt{\frac{1}{2}}$ by $(\sqrt{2}+3\sqrt{\frac{1}{2}})$ *Ans.* $\frac{1}{10}$.

R 4. Divide 1 by $\sqrt[4]{a}+\sqrt[4]{b}$.

Ans. $\frac{\sqrt[4]{a^3}-\sqrt[4]{a^2b}+\sqrt[4]{ab^2}-\sqrt[4]{b^3}}{a-b}$.

R 5. Divide $\sqrt[4]{a}+\sqrt[4]{b}$ by $\sqrt[4]{a}-\sqrt[4]{b}$.

Ans. $\frac{a+b+2\sqrt{ab}+2\sqrt[4]{a^3b}+2\sqrt[4]{ab^3}}{a-b}$.

R 6. Divide $\frac{43}{3}+\frac{13}{6}\sqrt{42}$ by $\left(\sqrt{\frac{7}{3}}+2\sqrt{\frac{1}{2}}\right)$.

Ans. $4\sqrt{\frac{7}{3}}+5\sqrt{\frac{1}{2}}$.

R **158. Powers of Radicals.**—Let $a\sqrt[n]{b}$ represent any radical of the n^{th} degree. Then we may raise this radical to the m^{th} power, by taking it m times as a factor; thus,

$$a\sqrt[n]{b}\times a\sqrt[n]{b}\times\ \ldots\ \times a\sqrt[n]{b}.$$

But, by the rule for multiplication, this continued product is equal to $a^m \sqrt[n]{b^m}$; whence,

$$(a\sqrt[n]{b})^m = a^m(\sqrt[n]{b^m}) \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

We have then, to raise a radical to any power, the following

RULE.

Raise the coefficient to the required power for a new coefficient, and the quantity under the radical sign to the required power for a new quantity under the radical sign, leaving the index unchanged.

EXAMPLES.

Raise the following radicals to the powers indicated:

1. $(\sqrt[4]{4a^3})^2$. *Ans* $2a\sqrt{a}$.

By the rule,

$$(\sqrt[4]{4a^3})^2 = \sqrt[4]{(4a^3)^2} = \sqrt[4]{16a^6} = 2a\sqrt[4]{a^2} = 2a\sqrt{a}, \text{ as above.}$$

2. $(3\sqrt[3]{2a})^5$. *Ans.* $486a\sqrt[3]{4a^2}$.

When the index of the radical is a multiple of the exponent of the power to which it is to be raised, the result can be simplified.

For, $\sqrt[4]{2a} = \sqrt{\sqrt{2a}}$ (Art. **151**); hence, in order to square $\sqrt[4]{2a}$, we have only to omit the first radical sign, which gives

$$(\sqrt[4]{2a})^2 = \sqrt{2a};$$

and similarly for other like cases; hence,

When the index of the radical is divisible by the exponent of the power to which it is to be raised, perform the division, leaving the quantity under the radical sign unchanged.

3. $(\sqrt[4]{a^2 - 2ab - c})^2$ *Ans.* $\sqrt{a^2 - 2ab - c}$.

4. $(3\sqrt[6]{4c^2})^2$. *Ans.* $9\sqrt[3]{4c^2}$.

5. $(\frac{1}{3}\sqrt[6]{3})^3$. *Ans.* $\frac{1}{27}\sqrt{3}$.

159. Extraction of Roots.—Let us take equation (1), of the last article,

$$(a\sqrt[n]{b})^m = a^m(\sqrt[n]{b^m}).$$

Extracting the m^{th} root of both members, we have

$$a\sqrt[n]{b} = \sqrt[m]{a^m \sqrt[n]{b^m}} = a\sqrt[n]{b},$$

since the second member must be equal to the first.

Hence, to extract any root of a radical of any degree, we have the following

RULE.

Extract the required root of the coefficient for a new coefficient, and the required root of the quantity under the radical sign for a new quantity under the radical sign, leaving the index unchanged.

EXAMPLES.

1. Find the cube root of $8\sqrt[4]{27}$. *Ans.* $2\sqrt[4]{3}$.

2. Find the fourth root of $\frac{1}{16}\sqrt[3]{256}$. *Ans.* $\frac{1}{2}\sqrt[3]{4}$.

When the indicated root of the quantity under the radical sign cannot be found, we may simplify the result by principle 3°, Art. **151**. Thus, to find the m^{th} root $c\sqrt[n]{d}$, we have, from the principle referred to,

$$\sqrt[m]{c\sqrt[n]{d}} = \sqrt[m]{c}\sqrt[mn]{d}.$$

Consequently, when we cannot extract the required root of the quantity under the radical sign, we have the following

RULE.

Extract the required root of the coefficient for a new coefficient, and multiply the given index by the index of the required root for a new index, leaving the quantity under the radical sign unchanged.

3. Find the cube root of $\sqrt[4]{3c}$. *Ans.* $\sqrt[12]{3c}$.

4. Find the square root of $\sqrt[3]{5c}$. *Ans.* $\sqrt[6]{5c}$.

When the quantity under the radical is a perfect power, of the *degree of either of the roots to be extracted*, the result can be simplified.

Thus, $\sqrt[3]{\sqrt[4]{8a^3}} = \sqrt[4]{\sqrt[3]{8a^3}} = \sqrt[4]{2a}$.

In like manner, $\sqrt{\sqrt[5]{9a^2}} = \sqrt[5]{\sqrt{9a^2}} = \sqrt[5]{3a}$.

5. Find the cube root of $\frac{1}{27}\sqrt{3}$. *Ans.* $\frac{1}{3}\sqrt[6]{3}$.

6. Find the cube root of $\frac{1}{8}\sqrt{2ab^2}$. *Ans.* $\frac{1}{2}\sqrt[6]{2ab^2}$.

160. Multiple Roots.—The rules just demonstrated depend on the principle, that *if two quantities are equal, the like roots of those quantities are also equal.*

This principle is true so long as we regard the term *root* in its general sense, but when the term is used in a restricted sense, it requires some modification. This modification is particularly necessary in operating on imaginary expressions, which are not roots, strictly speaking, but mere *indications of operations which it is impossible to perform.*

It has already been shown that every quantity has two square roots, equal with contrary signs. It may be shown it every quantity has *three cube roots, four fourth roots,* so on.

1°. Let a denote any quantity, and suppose p to be its arithmetical cube root; then will $a = p^3$. If we denote the algebraic value of the cube root of a, or of its equal p^3, by x, we shall have, from the definition of a cube root,

$$x^3 = p^3, \quad \text{or} \quad x^3 - p^3 = 0 \quad . \quad . \quad . \quad . \quad (1)$$

Factoring the first member of equation (1), we have,

$$(x - p)(x^2 + px + p^2) = 0 \quad . \quad . \quad . \quad . \quad (2)$$

Now, every value of x that will satisfy this equation will satisfy the first equation.. But this equation can be satisfied by supposing

$$x - p = 0, \quad \text{whence} \quad x = p;$$

or by supposing

$$x^2 + px + p^2 = 0,$$

from which we have,

$$x = -\frac{p}{2} \pm \frac{p}{2}\sqrt{-3}, \quad \text{and} \quad x = p\left(\frac{-1 \pm \sqrt{-3}}{2}\right);$$

hence, we see, that *there are three different algebraic cube roots of a, or of its equal p^3*, viz.:

$$p, \quad p\left(\frac{-1 + \sqrt{-3}}{2}\right) \quad \text{and} \quad p\left(\frac{-1 - \sqrt{-3}}{2}\right).$$

It will be shown in Art. **162**, that the cube of each of these expressions is equal to p^3.

2°. If we let p denote the arithmetical fourth root of a, and x its algebraic root, we have, as before,

$$x^4 = p^4, \quad \text{or} \quad x^4 - p^4 = 0 \quad . \quad . \quad . \quad . \quad (3)$$

Equation (3) may be placed under the form

$$(x^2 - p^2)(x^2 + p^2) = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equation (4) can be satisfied by supposing

$$x^2 - p^2 = 0; \qquad \text{whence} \qquad x = \pm p;$$

or by supposing

$$x^2 + p^2 = 0; \qquad \text{whence} \qquad x = \pm\sqrt{-p^2} = \pm p\sqrt{-1}.$$

Hence, *any quantity a has four algebraic fourth roots.*

3°. If p be the arithmetical sixth root of a, we have, as before,

$$x^6 = p^6, \qquad \text{or} \qquad x^6 - p^6 = 0 \quad . \quad . \quad . \quad . \quad (5)$$

Equation (5) can be placed under the form

$$(x^2 - p^2)(x^2 + px + p^2)(x^2 - px + p^2) = 0 \quad . \quad . \quad (6)$$

Equation (6) can be satisfied by supposing

$$x^2 - p^2 = 0; \qquad \text{or} \qquad x = \pm p;$$

or by supposing

$$x^2 + px + p^2 = 0;$$

$$\therefore \qquad x = -\frac{p}{2} \pm \sqrt{-p^2 + \frac{p^2}{4}} = p\left(\frac{-1 \pm \sqrt{-3}}{2}\right);$$

or by supposing

$$x^2 - px + p^2 = 0;$$

$$\therefore \qquad x = -\frac{p}{2} \pm \sqrt{-p^2 + \frac{p^2}{4}} = p\left(\frac{+1 \pm \sqrt{-3}}{2}\right).$$

Hence, *every quantity has six algebraic sixth roots.*

It may be demonstrated that any quantity has n algebraic n^{th} roots, n being any positive whole number. If n is even, and the quantity positive, two of the roots are *real* and equal, with contrary signs, and the others are imaginary. If the quantity is negative and n even, all of the roots are imaginary. But if n is odd, one of the roots is real and of the same sign as the quantity; all the others are imaginary.

161. If we make $a = 1$, whence $p = 1$, the results of the preceding article give us the various roots of 1.

We find in this manner, that the two square roots of 1 are

$$+1 \qquad \text{and} \qquad -1.$$

The three cube roots of 1 are

$$+1, \quad \frac{-1+\sqrt{-3}}{2}, \text{ and } \frac{-1-\sqrt{-3}}{2}.$$

The four fourth roots of 1 are

$$+1, \quad -1, \quad +\sqrt{-1}, \text{ and } -\sqrt{-1};$$

and so on.

Imaginary Quantities.

162. An **Imaginary Quantity** is an indicated even root of a negative quantity. Thus $\sqrt{-3}$, $\sqrt[4]{-2}$, and $\sqrt{-a^2}$, are imaginary quantities.

The rule for the multiplication of radicals, as explained in Art. **156**, requires some modification to adapt it to the case of imaginary quantities. By that rule the product of $+\sqrt{-4}$ and $+\sqrt{-3}$ is $+\sqrt{12}$, whereas, the real product is $-\sqrt{12}$, as will be seen hereafter.

Every imaginary monomial of the second degree appears under the form $\sqrt{-a^2}$. This may be resolved into two factors by principle 1°, Art. **151**, giving

$$\sqrt{-a^2} = \sqrt{a^2}\,\sqrt{-1} = \pm a\sqrt{-1}.$$

The factor $\sqrt{-1}$ is called the *imaginary factor*, and the other is called its *coefficient*.

When several factors of the form $\sqrt{-a^2}$ are to be multiplied together, we first reduce them to the form $\pm a\sqrt{-1}$. We then multiply their coefficients together by known rules, and the imaginary factors by a principle now to be deduced.

The first power of $\sqrt{-1}$ is $\sqrt{-1}$; the second power, by the definition of a square root, is -1; the third power is the product of the first and second powers, or $-1\sqrt{-1} = -\sqrt{-1}$; the fourth power is the square of the second, or $+1$; the fifth is the product of the first and fourth, that is, it is the same as the first; the sixth is the same as the second; the seventh, the same as the third; the eighth, the same as the fourth; the ninth, again, the same as the first; and so on indefinitely, as shown in the table, n being any whole number.

$$\begin{array}{rrcrr}
(\sqrt{-1})^1 = & \sqrt{-1}. & \qquad & (\sqrt{-1})^{4n+1} = & \sqrt{-1}. \\
(\sqrt{-1})^2 = & -1. & & (\sqrt{-1})^{4n+2} = & -1. \\
(\sqrt{-1})^3 = & -\sqrt{-1}. & & (\sqrt{-1})^{4n+3} = & -\sqrt{-1}. \\
(\sqrt{-1})^4 = & +1. & & (\sqrt{-1})^{4n+4} = & 1.
\end{array}$$

To show the use of this table, let it be required to find the continued product of $\sqrt{-4}$, $\sqrt{-3}$, $\sqrt{-2}$, $\sqrt{-7}$, and $\sqrt{-8}$. Reducing these expressions to the proper form, and indicating the multiplication, we have,

$$2\sqrt{-1} \times \sqrt{3}\sqrt{-1} \times \sqrt{2}\sqrt{-1} \times \sqrt{7}\sqrt{-1} \times 2\sqrt{2}\sqrt{-1}.$$

Changing the order of the factors,

$$(2 \times \sqrt{3} \times \sqrt{2} \times \sqrt{7} \times 2\sqrt{2})\,(\sqrt{-1})^5.$$

Hence, the product is $8\sqrt{21} \times \sqrt{-1}$, or, $8\sqrt{-21}$.

As an illustration of the foregoing principles, let it be required to verify the cube roots of a, as found in Article **160**. Cubing the coefficient $\frac{p}{2}$, we have $\frac{p^3}{8}$. Placing the other factor under the required form, we have,

$$-1 + \sqrt{3}\sqrt{-1}.$$

By the rule for multiplication, we have,

$$(-1+\sqrt{3}\,\sqrt{-1})(-1+\sqrt{3}\,\sqrt{-1})(-1+\sqrt{3}\,\sqrt{-1})=8.$$

Hence,

$$\left(\frac{p}{2}(-1+\sqrt{-3})\right)^3=p^3=a;$$

which was to be shown.

In like manner, all the other results of Art. **160** may be verified.

EXAMPLES.

Perform the operations indicated in the following examples:

1. $(a+b\sqrt{-1})\times(c+d\sqrt{-1})$.
Ans. $ac-bd+(ad+bc)\sqrt{-1}$.

2. $(a-b\sqrt{-1})\times(c-d\sqrt{-1})$.
Ans. $ac-bd-(ad+bc)\sqrt{-1}$.

3. $(a+b\sqrt{-1})^2$. *Ans.* $a^2-b^2+2ab\sqrt{-1}$.

4. $(a+b\sqrt{-1})^3$. *Ans.* $a^3-3ab^2+(3a^2b-b^3)\sqrt{-1}$.

5. $\sqrt{-a^2+2ab-b^2}$. *Ans.* $\pm(a-b)\sqrt{-1}$.

6. $\dfrac{1}{a+b\sqrt{-1}}+\dfrac{1}{a-b\sqrt{-1}}$. *Ans.* $\dfrac{2a}{a^2+b^2}$.

7. $\dfrac{a+b\sqrt{-1}}{a-b\sqrt{-1}}+\dfrac{a-b\sqrt{-1}}{a+b\sqrt{-1}}$. *Ans.* $\dfrac{2(a^2-b^2)}{a^2+b^2}$.

163. Let it be required to find the product of $\sqrt[4]{-a}$ by $\sqrt[4]{-b}$. By the ordinary rule for multiplication of radicals, we should find the product equal to $\sqrt[4]{ab}$; but this is not the true product. For, reducing the quantities to the required form, we have,

$$\sqrt[4]{a}\,\sqrt[4]{-1}\times\sqrt[4]{b}\,\sqrt[4]{-1}=\sqrt[4]{ab}\,[(\sqrt[4]{-1})^2]=\sqrt[4]{ab}\,\sqrt{-1}.$$

When we have to apply the rule for the multiplication of radicals to imaginary quantities of the fourth degree, we first reduce the imaginary terms to the form $\sqrt[4]{a}\sqrt[4]{-1}$, and then proceed as in the above example.

VI. FRACTIONAL EXPONENTS.

164. It has been agreed to denote radicals by means of *fractional exponents.* In this system of notation, the index of the radical forms the denominator of the exponent. When a simple radical is to be denoted, the numerator of the exponent is 1; when the quantity under the radical sign is raised to a power, the numerator of the exponent is the same as the exponent of the power. Thus, $\sqrt[3]{a}$ is denoted by the symbol $a^{\frac{1}{3}}$; $\sqrt[4]{a^3}$, by the symbol $a^{\frac{3}{4}}$; $\sqrt[m]{a^n}$, by the symbol $a^{\frac{n}{m}}$; and so on.

165. If we combine this method of notation with that already explained, for denoting reciprocals, we have the following table of equivalent expressions:

TABLE.

1°.	$\sqrt[n]{a}$		equivalent to	$a^{\frac{1}{n}}$
2°.	$\sqrt[n]{a^m}$	or $(\sqrt[n]{a})^m$	"	$a^{\frac{m}{n}}$.
3°.	$\frac{1}{a^n}$		"	a^{-n}.
4°.	$\frac{1}{\sqrt[n]{a}}$	or $\sqrt[n]{\frac{1}{a}}$	"	$a^{-\frac{1}{n}}$.
5°.	$\frac{1}{\sqrt[n]{a^m}}$	or $\sqrt[n]{\frac{1}{a^m}}$	"	$a^{-\frac{m}{n}}$.

166. We see from the 3° principle of the above table that,

$$\frac{1}{a^n} = a^{-n} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

Taking the reciprocal of both members of this equation, we have,

$$a^n = \frac{1}{a^{-n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From equations (1) and (2), we see that *a factor may be transferred from the numerator to the denominator, or from the denominator to the numerator, by changing the sign of its exponent.*

167. It may be shown that all the rules for operating on quantities whose *exponents* are whole numbers, are applicable when the exponents are fractional.

In the first place, it is plain that both numerator and denominator of a fractional exponent may be multiplied by the same quantity without altering the value of the expression; for, by definition, the m^{th} power of the m^{th} root of a quantity is equal to the quantity itself. This principle enables us to reduce quantities, having fractional exponents, to equivalent ones in which the exponents have a common denominator.

Thus, the quantities $a^{\frac{m}{n}}$ and $a^{\frac{r}{s}}$ are equivalent to the quantities $a^{\frac{ms}{ns}}$ and $a^{\frac{nr}{ns}}$. But the latter expressions have a common index. Hence, *radicals denoted by means of fractional exponents may be reduced to equivalent radicals having a common index, by simply reducing their exponents to a common denominator.*

Let $a^{\frac{m}{n}}$ and $a^{\frac{r}{n}}$ be any two radicals after they are reduced to a common index. Their product by the rule for exponents is

$$a^{\frac{m}{n}} \times a^{\frac{r}{n}} = a^{\frac{m+r}{n}}.$$

But, we have, by using equivalent expressions,

$$\sqrt[n]{a^m} \times \sqrt[n]{a^r} = \sqrt[n]{a^{m+r}} = a^{\frac{m+r}{n}}.$$

If one exponent is + and the other −, we have, by the rule for exponents,

$$a^{\frac{m}{n}} \times a^{\frac{-r}{n}} = a^{\frac{m-r}{n}}.$$

But, by using equivalent expressions, we have

$$\sqrt[n]{a^m} \times \frac{1}{\sqrt[n]{a^r}} = \sqrt[n]{\frac{a^m}{a^r}} = \sqrt[n]{a^{m-r}} = a^{\frac{m-r}{n}}.$$

If both exponents are negative, we have, by the rule for exponents,

$$a^{-\frac{m}{n}} \times a^{-\frac{r}{n}} = a^{-\frac{m+r}{n}}.$$

But, by using equivalent expressions, we have

$$\frac{1}{\sqrt[n]{a^m}} \times \frac{1}{\sqrt[n]{a^r}} = \frac{1}{\sqrt[n]{a^{m+r}}} = a^{-\frac{m+r}{n}}.$$

Hence, for the multiplication of quantities whose exponents are fractional, either positive or negative, we have the same rule as though they were whole numbers. And because division is the reverse of multiplication, the rule for exponents will be the same whether the exponents are entire or fractional. Hence, it follows, from what precedes, that the methods of forming powers and extracting roots will be the same whether the exponents are positive or negative, entire or fractional.

EXAMPLES.

Perform the operations indicated below:

1. $a^{\frac{3}{4}}b^{-\frac{1}{2}}c^{-1} \times a^2b^{\frac{2}{3}}c^{\frac{2}{5}}$. *Ans.* $a^{\frac{11}{4}}b^{\frac{1}{6}}c^{-\frac{3}{5}}$.

2. $3a^{-2}b^{\frac{2}{3}} \times 2a^{-\frac{4}{5}}b^{\frac{1}{2}}c^2$. *Ans.* $6a^{-\frac{14}{5}}b^{\frac{7}{6}}c^2$.

3. $6a^{-\frac{1}{2}}b^4c^{-m} \times 5a^{\frac{1}{3}}b^{-5}c^n$. *Ans.* $30a^{-\frac{1}{6}}b^{-1}c^{n-m}$.

4. $\left(\frac{2}{3}a^{\frac{1}{3}}\right)^2$ *Ans.* $\frac{4}{9}a^{\frac{2}{3}}$.

5. $\left(\frac{1}{3}a^{\frac{1}{2}}\right)^3$ *Ans.* $\frac{1}{27}a^{\frac{3}{2}}$.

6. $a^{\frac{2}{3}} \div a^{-\frac{3}{4}}$. *Ans.* $a^{\frac{17}{12}}$.

7. $a^{\frac{3}{4}} \div a^{\frac{4}{5}}$. *Ans.* $a^{\frac{3}{4}-\frac{4}{5}} = a^{-\frac{1}{20}}$.

8. $a^{\frac{2}{5}} \times b^{\frac{3}{4}} \div a^{-\frac{1}{2}}b^{\frac{7}{8}}$. *Ans.* $a^{\frac{9}{10}}b^{-\frac{1}{8}}$.

9. $32a^{\frac{1}{2}}b^6c^{\frac{5}{2}} \div 8a^{\frac{1}{6}}b^5c^{-\frac{3}{2}}$. *Ans.* $4a^{\frac{1}{3}}bc^4$.

10. $64a^9b^{\frac{7}{2}}c^{-\frac{3}{5}} \div 32a^{-9}b^{-\frac{3}{2}}c^{-\frac{3}{5}}$. *Ans.* $2a^{18}b^5$.

11. $\sqrt[3]{a^{\frac{2}{3}}} \times \sqrt[4]{b^{\frac{4}{3}}} \times \sqrt{c^{\frac{3}{2}}}$. *Ans.* $a^{\frac{2}{9}}b^{\frac{1}{3}}c^{\frac{3}{4}}$.

12. $\dfrac{2\sqrt{2} \times (3)^{\frac{1}{3}}}{\frac{1}{2}\sqrt{2}}$ *Ans.* $4\sqrt[3]{3}$.

13. $\left\{\dfrac{\frac{1}{2}(2)^{\frac{1}{2}}\sqrt[3]{3}}{2\sqrt[4]{2}\,(3)^{\frac{1}{2}}}\right\}^4$ *Ans.* $\frac{1}{384}\sqrt[3]{3}$.

14. $\sqrt{\left\{\dfrac{(\frac{1}{2})^3 + \sqrt{3\frac{1}{2}}}{2\sqrt{2}\,.\,(\frac{3}{4})^{\frac{1}{2}}}\right\}^{\frac{1}{2}}}$ *Ans.* $\sqrt[4]{\frac{1}{6}\left(\frac{1}{8}\sqrt{6} + \sqrt{21}\right)}$.

15. What is the product of

$$a^{\frac{5}{2}} + a^2b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} + ab + a^{\frac{1}{2}}b^{\frac{4}{3}} + b^{\frac{5}{3}} \text{ by } a^{\frac{1}{2}} - b^{\frac{1}{3}}.$$

OPERATION.

$$\begin{array}{l}
a^{\frac{5}{2}} + a^2b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} + ab + a^{\frac{1}{2}}b^{\frac{4}{3}} + b^{\frac{5}{3}} \\
a^{\frac{1}{2}} - b^{\frac{1}{3}} \\
\hline
a^3 + a^{\frac{5}{2}}b^{\frac{1}{3}} + a^2b^{\frac{2}{3}} + a^{\frac{3}{2}}b + ab^{\frac{4}{3}} + a^{\frac{1}{2}}b^{\frac{5}{3}} \\
\qquad - a^{\frac{5}{2}}b^{\frac{1}{3}} - a^2b^{\frac{2}{3}} - a^{\frac{3}{2}}b - ab^{\frac{4}{3}} - a^{\frac{1}{2}}b^{\frac{5}{3}} - b^3 \\
\hline
a^3 - b^3. \quad \textit{Ans.}
\end{array}$$

16. Divide $a^{\frac{7}{3}} - a^2b^{-\frac{2}{3}} - a^{\frac{1}{3}}b + b^{\frac{1}{3}}$ by $a^{\frac{1}{3}} - b^{-\frac{2}{3}}$.

Ans. $a^2 - b$.

R 17. Divide $x^{\frac{3}{4}} - \frac{3}{4}a^{\frac{3}{4}}$ by $x^{\frac{1}{2}} - \frac{1}{2}a^{\frac{1}{2}}$.

Ans. $x^{\frac{1}{4}} + \frac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{4}} + \frac{1}{4}ax^{-\frac{3}{4}} +$ &c.

18. Multiply $x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$ by $y + x^{\frac{1}{2}}y^{\frac{1}{2}} + x$.

Ans. $x^{-1}y + 2x^{-\frac{1}{2}}y^{\frac{1}{2}} + 3 + 2x^{\frac{1}{2}}y^{-\frac{1}{2}} + xy^{-1}$.

19. Find the square of $\dfrac{a^{\frac{2}{3}}}{2} - \dfrac{b^{\frac{2}{3}}}{3}$.

Ans. $\frac{1}{4}a^{\frac{4}{3}} - \frac{1}{3}a^{\frac{2}{3}}b^{\frac{2}{3}} + \frac{1}{9}b^{\frac{4}{3}}$.

20. Multiply $x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$ by $x^{-1} - x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$.

Ans. $x^{-2} + x^{-1}y^{-1} + y^{-2}$.

21. Find the square of $x^{-\frac{1}{2}} - y^{-\frac{1}{2}}$.

Ans. $x^{-1} - 2x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$.

22. Multiply $x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + 2$ by $x - x^{\frac{1}{2}}$.

Ans. $x^{\frac{5}{2}} - x^2 - 4x^{\frac{3}{2}} + 6x - 2x^{\frac{1}{2}}$.

R 23. Divide $\frac{1}{5}x^{\frac{8}{15}} - \frac{3}{10}x^{\frac{7}{10}} + \frac{1}{2}x^{\frac{3}{4}} - \frac{1}{3}x^{\frac{7}{12}} - \frac{3}{20}x^{\frac{9}{20}} + \frac{1}{4}x^{\frac{1}{2}}$ by $x^{\frac{1}{4}} - \frac{3}{5}x^{\frac{1}{5}}$.

Ans. $\frac{1}{2}x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{1}{3}} + \frac{1}{4}x^{\frac{1}{4}}$.

24. Multiply $x^2 + xy + y^2$ by $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.

Ans. $x^3 + x^{\frac{5}{2}}y^{\frac{1}{2}} + 2x^2y + x^{\frac{3}{2}}y^{\frac{3}{2}} + 2xy^2 + x^{\frac{1}{2}}y^{\frac{5}{2}} + y^3$.

25. Multiply $x^{\frac{2}{3}} + 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} + 2x^{\frac{1}{6}} + 1$ by $x^{\frac{1}{3}} - 2x^{\frac{1}{6}} + 1$.

Ans. $x - 2x^{\frac{1}{2}} + 1$.

R 26. Extract the square root of $x^2 + x^{\frac{3}{2}} - 1\frac{1}{12}x - \frac{2}{3}x^{\frac{1}{2}} + \frac{4}{9}$.

Ans. $x + \frac{1}{2}x^{\frac{1}{2}} - \frac{2}{3}$.

CHAPTER VIII.

FORMATION, PROPERTIES, AND SUMMATION OF SERIES.

I. ARITHMETICAL PROGRESSION.

168. A Series is a succession of terms, each of which is derived from one or more of the preceding ones by a *fixed law*. This law is called the *law of the series.*

169. An Arithmetical Progression is a series, in which each term is derived from the preceding one by the addition of a constant quantity called the *common difference.*

If the common difference is *positive,* each term is greater than the preceding one, and the progression is said to be *increasing.*

If the common difference is *negative,* each term is less than the preceding one, and the progression is said to be *decreasing.*

Thus, ... 1, 3, 5, 7, . . . &c., is an *increasing arithmetical progression,* in which the common difference is 2; and . . . . . 19, 16, 13, 10, 7, . . . is a *decreasing arithmetical progression,* in which the common difference is — 3.

170. Names of Terms.—When a certain number of terms of an arithmetical progression is considered, the first of these is called the *first term of the progression,* the last is called the *last term of the progression,* and both together are called the *extremes.* All the terms between the extremes are called *arithmetical means.* An arithmetical progression is often called a *progression by differences.*

171. To Find any Term.—Let d represent the common difference of the arithmetical progression,

$$a \,.\, b \,.\, c \,.\, e \,.\, f \,.\, g \,.\, h \,.\, k, \text{ \&c.,}$$

which is written by placing a period between each two of the terms.

From the definition of a progression, it follows that,

$$b = a + d, \quad c = b + d = a + 2d, \quad e = c + d = a + 3d;$$

and, in general, any term of the series is equal to *the first term plus as many times the common difference as there are preceding terms.*

Thus, let l be any term, and n the number that marks its place. Then, the number of preceding terms will be $n - 1$, and the expression for this *general term* will be

$$l = a + (n - 1)d.$$

If d is positive, the progression is increasing; hence,

In an increasing arithmetical progression, any term is equal to the first term, plus the product of the common difference by the number of preceding terms.

If we make $n = 1$, we have $l = a$; in this case, there is but one term.

If we make $n = 2$, we have $l = a + d$; that is, there are two terms, and the second is equal to the first plus the common difference.

EXAMPLES.

1. If $a = 3$ and $d = 2$, what is the 3d term? *Ans.* 7.

2. If $a = 5$ and $d = 4$, what is the 6th term? *Ans.* 25.

3. If $a = 7$ and $d = 5$, what is the 9th term? *Ans.* 47.

The formula, $l = a + (n - 1)\,d,$

enables us to find any term without determining those that precede it.

To find the 50th term of the progression,

$$1 \,.\, 4 \,.\, 7 \,.\, 10 \,.\, 13 \,.\, 16 \,.\, 19, \,.\,.$$

we have,

$$l = 1 + 49 \times 3 = 148.$$

For the 60th term of the progression,

$$1 \,.\, 5 \,.\, 9 \,.\, 13 \,.\, 17 \,.\, 21 \,.\, 25, \,.\,.\,.$$

we have,

$$l = 1 + 59 \times 4 = 237.$$

172. Decreasing Progression.—If d is negative, the progression is decreasing, and the formula becomes

$$l = a + (n - 1) \times . - d = a - (n - 1)\,d; \text{ that is,}$$

Any term of a decreasing arithmetical progression is equal to the first term minus the product of the common difference by the number of preceding terms.

EXAMPLES.

1. The first term of a decreasing progression is 60, and the common difference -3: what is the 20th term?

We have $l = 60 - (20 - 1)\,3 = 60 - 57 = 3.$ *Ans.*

2. The first term is 90, the common difference -4: what is the 15th term? *Ans.* 34.

3. The first term is 100, and the common difference -2: what is the 40th term? *Ans.* 22.

173. Sum of Extremes.—If we take an arithmetical progression,

$$a \,.\, b \,.\, c \,.\,.\,.\,.\,.\, i \,.\, k \,.\, l,$$

having n terms, and a common difference d, and designate the term which has p terms before it, by t, we shall have

$$t = a + pd \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

If we revert the order of terms of the progression, considering l as the first term, we shall have a new progression whose common difference is $-d$. The term of this new progression which has p terms before it will evidently be the same as that which has p terms after it in the given progression, and if we represent that term by t', we shall have

$$t' = l - pd \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Adding equations (1) and (2), member to member, we find

$$t + t' = a + l; \qquad \text{hence,}$$

The sum of any two terms, at equal distances from the extremes, is equal to the sum of the extremes.

174. Sum of Terms.—If the sum of the terms of a progression be represented by S, and a new progression be formed, by reversing the order of its terms, we shall have

$$S = a + b + c + \; . \; . \; . \; . \; + i + k + l,$$
$$S = l + k + i + \; . \; . \; . \; . \; + c + b + a,$$

Adding these equations, member to member, we get

$$2S = (a+l) + (b+k) + (c+i) \ldots + (i+c) + (k+b) + (l+a);$$

and, since the sums, $a + l$, $b + k$, $c + i$ . . . . are equal to each other, and their number equal to n, we have

$$2S = (a + l)n, \quad \text{or} \quad S = \left(\frac{a+l}{2}\right)n; \qquad \text{that is,}$$

The sum of the terms of an arithmetical progression is equal to half the sum of the two extremes multiplied by the number of terms.

EXAMPLES.

1. The extremes are 2 and 16, and the number of terms 8: what is the sum of the series?

$$S = \left(\frac{a+l}{2}\right) \times n \quad \text{gives} \quad S = \frac{2+16}{2} \times 8 = 72.$$

2. The extremes are 3 and 27, and the number of terms 12: what is the sum of the series? *Ans.* 180.

3. The extremes are 4 and 20, and the number of terms 10: what is the sum of the series? *Ans.* 120.

4. The extremes are 8 and 80, and the number of terms 10: what is the sum of the series? *Ans.* 440.

175. General Formulas.—The formulas

$$l = a + (n - 1)\,d \qquad \text{and} \qquad S = \left(\frac{a + l}{2}\right) \times n$$

contain five quantities, a, d, n, l, and S, and consequently, when any three of them are given, the other two can be found.

The different cases that may arise are shown in the following

TABLE.

No.	Given.	Unknown.	Values of the Unknown Quantities.
1	a, d, n	l, S	$l=a+(n-1)d$; $S=\frac{1}{2}n\,[2a+(n-1)\,d]$.
2	a, d, l	n, S	$n=\frac{l-a}{d}+1$; $S=\frac{(l+a)\,(l-a+d)}{2d}$.
3	a, d, S	n, l	$n=\frac{d-2a\pm\sqrt{(d-2a)^2+8dS}}{2d}$; $l=a+(n-1)d$.
4	a, n, l	S, d	$S=\frac{1}{2}n\,(a+l)$; $d=\frac{l-a}{n-1}$.
5	a, n, S	d, l	$d=\frac{2\,(S-an)}{n\,(n-1)}$; $l=\frac{2\,S}{n}-a$.
6	a, l, S	n, d	$n=\frac{2\,S}{a+l}$; $d=\frac{(l+a)\,(l-a)}{2\,S-(l+a)}$.
7	d, n, l	a, S	$a=l-(n-1)d$; $S=\frac{1}{2}n\,[2l-(n-1)\,d]$.
8	d, n, S	a, l	$a=\frac{2\,S-n\,(n-1)\,d}{2n}$; $l=\frac{2\,S+n\,(n-1)\,d}{2n}$.
9	d, l, S	n, a	$n=\frac{2l+d\pm\sqrt{(2l+d)^2-8dS}}{2d}$; $a=l-(a-1)\,d$.
10	n, l, S	a, d	$a=\frac{2\,S}{n}-l$; $d=\frac{2\,(nl-S)}{n\,(n-1)}$.

176. From the formula

$$l = a + (n - 1)d,$$

we have

$$a = l - (n - 1)d; \qquad \text{that is,}$$

The first term of an increasing arithmetical progression is equal to any following term, minus the product of the common difference by the number of preceding terms.

177. From the same formula, we also find

$$d = \frac{l - a}{n - 1}; \qquad \text{that is,}$$

In any arithmetical progression, the common difference is equal to the last term, minus the first divided by the number of terms less one.

If the last term is less than the first, the common difference is negative, as it should be.

EXAMPLES.

1. The first term of a progression is 4, the last term 16, and the number of terms considered 5: what is the common difference?

The formula

$$d = \frac{l - a}{n - 1} \quad \text{gives} \quad d = \frac{16 - 4}{4} = 3.$$

2. The first term of a progression is 22, the last term 4, and the number of terms considered 10: what is the common difference? *Ans.* -2.

178. Problems.—By the aid of the last principle, we can solve the following problem, viz.:

To insert m of arithmetical means between two given numbers a and b.

To solve this problem, it is necessary to find the common difference. Now, we may regard a as the first term of an arithmetical progression, b as the last term, and the required means as intermediate terms. The number of terms of this progression will be $m + 2$.

Now, by substituting in the above formula, b for l, and $m + 2$ for n, it becomes

$$d = \frac{b - a}{m + 2 - 1}, \quad \text{or} \quad d = \frac{b - a}{m + 1};$$

that is, *the common difference* is obtained by dividing the difference between the last and first terms by the required number of means plus 1.

Having found the common difference, the *first arithmetical mean* is formed by adding d, or $\frac{b - a}{m + 1}$, to the first term, a. The *second mean* is obtained by augmenting the first, by d, etc.

EXAMPLES.

1. Insert 3 arithmetical means between 2 and 18. The formula

$$d = \frac{b - a}{m + 1}, \quad \text{gives} \quad d = \frac{18 - 2}{4} = 4;$$

hence, the progression is

$$2 \;.\; 6 \;.\; 10 \;.\; 14 \;.\; 18.$$

2. Insert 12 arithmetical means between 77 and 12. The formula

$$\frac{b - a}{m + 1}, \quad \text{gives} \quad d = \frac{13 - 77}{13} = -5;$$

hence, the progression is

$$77 \,.\, 72 \,.\, 67 \,.\, 62 \,.\,.\,.\,.\,.\, 22 \,.\, 17 \,.\, 12.$$

3. Insert 9 arithmetical means, between 75 and 5.

Ans. Progression 75 . 68 . 61 . . . . . 26 . 19 . 12 . 5.

179. If the same number of arithmetical means be inserted between each pair of consecutive terms of a progression, these terms and the arithmetical means, taken together, will form a progression.

For, let $a \,.\, b \,.\, c \,.\, e \,.\, f \ldots.$ be the given progression, and m the number of means to be inserted between a and b, b and c, c and $e \ldots..$

From what has just been said, the common differences of the partial progressions are

$$\frac{b-a}{m+1}, \quad \frac{c-b}{m+1}, \quad \frac{e-c}{m+1} \ldots.$$

and these are equal to each other, since, a, b, c, ... are in progression: therefore, the common difference is the same in each partial progression; and since the *last term* of the first forms the *first term* of the second, etc., we conclude that all of the partial progressions form a single progression.

GENERAL EXAMPLES.

1. Find the sum of the first fifty terms of the progression

$$2 \,.\, 9 \,.\, 16 \,.\, 23 \ldots$$

For the 50th term, we have

$$l = 2 + 49 \times 7 = 345.$$

Hence,

$$S = (2 + 345) \times \frac{50}{2} = 347 \times 25 = 8675.$$

2. Find the 100th term of the series $2 \,.\, 9 \,.\, 16 \,.\, 23 \ldots$

Ans. 695.

3. Find the sum of 100 terms of the series $1 \,.\, 3 \,.\, 5 \,.\, 7 \,.\, 9 \ldots$

Ans. 10000.

4. The greatest term is 70, the common difference 3, and the number of terms 21: what is the least term and the sum of the terms?

Ans. Least term 10; sum of terms 840.

5. The first term of a decreasing arithmetical progression is 10, the common differance $-\frac{1}{3}$, and the number of terms 21: required the sum of the terms. *Ans.* 140.

6. In a progression by differences, having given the common difference 6, the last term 185, and the sum of the terms 2945: find the first term, and the number of terms.

Ans. First term $= 5$; number of terms 31.

7. Find 9 arithmetical means between each antecedent and consequent of the progression $2 . 5 . 8 . 11 . 14 \ldots$

Ans. $d = 0.3$.

8. Find the number of men contained in a triangular battalion, the first rank containing 1 man, the second 2, the third 3, and so on to the n^{th}, which contains n. In other words, find the expression for the sum of the natural numbers $1, 2, 3, \ldots$ from 1 to n, inclusively.

Ans. $S = \dfrac{n(n+1)}{2}$.

9. Find the sum of the first n terms of the progression of uneven numbers $1, 3, 5, 7, 9 \ldots$ *Ans.* $S = n^2$.

10. One hundred stones being placed on the ground, in a straight line, at the distance of two yards from each other, how far will a person travel who shall bring them one by one to a basket, placed at two yards from the first stone?

Ans. 11 miles 840 yards.

II. OF RATIO AND PROPORTION.

180. The **Ratio** of one quantity to another, is the quotient that arises from dividing the second by the first. Thus, the ratio of a to b is $\dfrac{b}{a}$. Here, a is regarded as the *unit*, or *standard*, and b as the quantity measured.

181. *Two* quantities are said to be proportional when their ratio remains the same, while the quantities themselves change in value. Thus, if the ratio of a to b remains the same, while a and b undergo changes in value, then a is said to be proportional to b. Thus, $a \propto b$.

182. *Four* quantities are in *proportion* when the ratio of the first to the second is equal to the ratio of the third to the fourth.

Thus, if

$$\frac{b}{a}=\frac{d}{c},$$

the quantities a, b, c, and d, are said to be in proportion. We may indicate that these quantities are proportional by writing them as follows:

$$a : b :: c : d.$$

This expression is read, a is to b, as c is to d, and it is called *a proportion.*

183. The quantities compared are called *terms* of the proportion.

The first and fourth terms are called *extremes,* the second and third are called *means;* the first and third are called *antecedents,* the second and fourth are called *consequents,* and the fourth is said to be a *fourth proportional* to the other three.

If the second and third terms are the same, either is said to be a *mean proportional* between the other two. Thus, in the proportion

$$a : b :: b : c,$$

b is a mean proportional between a and c, and c is said to be a third proportional to a and b.

184. Two quantities are reciprocally proportional when one is proportional to the reciprocal to the other.

Thus, if the product of x and y is equal to 1, we have

$$xy=1, \quad \text{whence,} \quad x=\frac{1}{y} \quad \text{and} \quad y=\frac{1}{x}.$$

In these equations, if x increases, y diminishes in the same ratio, and the reverse; also, x increases with the reciprocal of y, and y with the reciprocal of x.

III. GEOMETRICAL PROGRESSION.

185. A Geometrical Progression is a series in which each term is derived from the preceding by multiplying it by a constant quantity called the *ratio of the progression.*

When the ratio is greater than 1, each term is greater than the preceding, and the progression is said to be *increasing.* Thus,

$$\ldots 3 : 6 : 12 : 24 : 48 \ldots .$$

is an increasing progression, whose ratio is 2.

When the ratio is less than 1, each term is less than the preceding, and the progression is said to be *decreasing.* Thus,

$$\ldots 16 : 8 : 4 : 2 : 1 : \tfrac{1}{2} : \ldots .$$

is a decreasing progression, whose ratio is $\frac{1}{2}$.

A *geometrical progression* is a continued proportion, in which each term is a mean proportional between the preceding and succeeding terms.

186. Any Term.—Let r denote the ratio of the progression

$$\ldots a : b : c : d : e \ldots .$$

From the definition, we have

$$\ldots b = ar, \quad c = br = ar^2, \quad d = cr = ar^3, \quad e = dr = ar^4 \ldots;$$

and in general, the n^{th} term, that is, the term that has $n-1$ terms before it, is equal to ar^{n-1}. Denoting this term by l, we have the formula

$$l = ar^{n-1}$$

Hence, *any term of a geometrical progression is equal to the first term multiplied by the ratio raised to a power whose exponent is equal to the number of preceding terms.*

By means of this principle, we may find any term of a progression without finding the preceding terms.

EXAMPLES.

1. Find the 5th term of the progression

$$2 : 4 : 8 : 16, \text{ \&c.},$$

in which the first term is 2, and the common ratio 2.

$$\text{5th term } = 2 \times 2^4 = 2 \times 16 = 32.$$

2. Find the 8th term of the progression

$$2 : 6 : 18 : 54 \ldots$$

$$\text{8th term } = 2 \times 3^7 = 2 \times 2187 = 4374.$$

3. Find the 12th term of the progression

$$64 : 16 : 4 : 1 : \frac{1}{4} \,.\,.$$

$$\text{12th term } = 64\left(\frac{1}{4}\right)^{11} = \frac{4^3}{4^{11}} = \frac{1}{4^8} = \frac{1}{65536}.$$

187. Sum of Terms.—We will now explain the method of determining the sum of n terms of the progression

$$a : b : c : d : e : f : \ldots : i : k : l,$$

whose ratio is r.

If we denote the sum of the series by S, and the n^{th} term by l, we shall have

$$S = a + ar + ar^2 \ldots . + ar^{n-2} + ar^{n-1} \quad . \quad . \quad (1)$$

Multiplying both members by r, we have

$$Sr = ar + ar^2 + ar^3 \ldots + ar^{n-1} + ar^n \quad . \quad . \quad (2)$$

Subtracting equation (1) from (2), member from member, we have

$$Sr - S = ar^n - a; \qquad \text{whence} \qquad S = \frac{ar^n - a}{r - 1};$$

substituting for ar^n, its value lr, we have

$$S = \frac{lr - a}{r - 1}.$$

Hence, to find the sum of any number of terms of a geometrical progression, we have the following

RULE.

Multiply the last term by the ratio, subtract the first term from the product, and divide the remainder by the ratio less 1.

EXAMPLES.

1. Find the sum of eight terms of the progression

$$2 : 6 : 18 : 54 : 162 \ldots : 4374.$$

$$S = \frac{lr - a}{r - 1} = \frac{13122 - 2}{2} = 6560.$$

2. Find the sum of five terms of the progression

$$2 : 4 : 8 : 16 : 32 ; \ldots .$$

$$S = \frac{lr - a}{r - 1} = \frac{64 - 2}{1} = 62. \quad \textit{Ans.}$$

3. Find the sum of ten terms of the progression

$$2 : 6 : 18 : 54 : 162 \ldots 2 \times 3^9 = 39366.$$

Ans. 59048.

4. What debt may be discharged in twelve months, by paying $1 the first month, $2 the second month, $4 the third month, and so on, each succeeding payment being double the last; and what will be the last payment?

Ans. Debt, $4095; last payment, $2048.

5. A gentleman married his daughter on New-Year's day and gave her husband 1*s.* toward her portion, and was to double it on the first day of every month during the year: what was her portion? *Ans.* £204 15*s.*

6. A man bought 10 bushels of wheat on condition that he should pay 1 cent for the first bushel, 3 for the second, 9 for the third, and so on to the last: what did he pay for the last bushel, and for the ten bushels?

Ans. Last bushel, \$196.83; total cost, \$295.24.

188. When the progression is decreasing, we have $r < 1$ and $l < a$; the formula for the sum may then be written under the form

$$S = \frac{a - lr}{1 - r},$$

in which both terms of the fraction will be positive.

Substituting ar^{n-1} for l, in the expression for S, we have

$$S = \frac{ar^n - a}{1 - r} \quad \text{or} \quad S = \frac{a - ar^n}{-1 + r}.$$

EXAMPLES.

1. Find the sum of five terms of the progression

$$32 : 16 : 8 : 4 : 2.$$

$$S = \frac{a - lr}{1 - r} = \frac{32 - 2 \times \frac{1}{2}}{\frac{1}{2}} = \frac{31}{\frac{1}{2}} = 62.$$

2. Find the sum of twelve terms of the progression

$$64 : 16 : 4 : 1 : \frac{1}{4} \ldots . : \frac{1}{65536}.$$

$$S = \frac{a - lr}{1 - r} = \frac{64 - \frac{1}{65536} \times \frac{1}{4}}{\frac{3}{4}} = \frac{256 - \frac{1}{65536}}{3} = 85 + \frac{65535}{196608}.$$

We perceive that the principal difficulty consists in obtaining the numerical value of the last term, a tedious operation, even when the number of terms is not very great.

189. If in the formula

$$S = \frac{a(r^n - 1)}{r - 1},$$

we make $r = 1$, it reduces to

$$S = \frac{0}{0}.$$

This result arises from the existence of a common factor, $r - 1$, in both terms of the fraction, which factor becomes 0, under the particular supposition.

If we divide out this factor (Art. **80**), we find,

$$S = a\,(r^{n-1} + r^{n-2} + r^{n-3} + \ldots + r + 1),$$

in which, if we make $r = 1$, we have

$$S = a\,(1 + 1 + \ldots + 1) = na.$$

This result corresponds to the hypothesis; for, under the supposition made, each term of the progression became equal to a; and since there are n of them, their sum should be na.

190. From the two formulas

$$l = ar^{n-1}, \quad \text{and} \quad S = \frac{lr - a}{r - 1},$$

several properties may be deduced. We shall consider only some of the most important.

The first formula gives

$$r^{n-1} = \frac{l}{a}, \quad \text{whence,} \quad r = \sqrt[n-1]{\frac{l}{a}}.$$

The expression

$$r = \sqrt[n-1]{\frac{l}{a}},$$

furnishes the means for solving the following problem, viz:

To find m geometrical means between two given numbers a and b; that is, *to find m means, which will form with a and b, considered as extremes, a geometrical progression.*

To find this series, it is necessary to know the ***ratio.*** Now, the number of means being m, the total number of terms will be $m + 2$. Moreover, we have $l = b$; therefore, the value of r is

$$r = \sqrt[m+1]{\frac{b}{a}}; \text{ that is,}$$

To find the ratio, divide the second number by the first; then extract that root of the quotient whose index is one greater than the number of required means:

Hence, the progression is

$$a : a\sqrt[m+1]{\frac{b}{a}} : a\sqrt[m+1]{\frac{b^2}{a^2}} : a\sqrt[m+1]{\frac{b^3}{a^3}} \dots . b.$$

EXAMPLES.

1. To insert six geometrical means between the numbers 3 and 384, we make $m = 6$, whence from the formula,

$$= \sqrt[7]{\frac{384}{3}} = \sqrt[7]{128} = 2;$$

hence the progression

$$3 : 6 : 12 : 24 : 48 : 96 : 192 : 384.$$

2. Insert four geometrical means between 2 and 486. The progression is

$$2 : 6 : 18 : 54 : 162 : 486.$$

When the same number of geometrical means are inserted between each two of the terms of a geometrical progression, all the partial progressions thus formed will, when taken together, constitute a single progression.

191. Infinite Number of Terms.—Let there be a decreasing progression

$$a : b : c : d : e : f : \ldots,$$

containing an infinite number of terms. The formula

$$S = \frac{a - ar^n}{1 - r},$$

which expresses the sum of n terms, can be put under the form

$$S = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

Now, since the progression is decreasing, r is a proper fraction, and r^n is also a fraction, which diminishes as n increases. Therefore, the greater number of terms we take, the more will $\frac{a}{1 - r} \times r^n$ diminish, and consequently, the nearer will the sum of these terms approximate to an equality with the first part of S; that is, to $\frac{a}{1 - r}$. Finally, when n is taken greater than any assignable number, or when

$$n = \infty, \quad \text{then} \quad \frac{a}{1 - r} \times r^n$$

will be less than any assignable number, or will be denoted by 0; and the expression $\frac{a}{1 - r}$ will represent the true value of the sum of all the terms of the series. Hence,

The sum of the terms of a decreasing progression, in which the number of terms is infinite, is

$$S = \frac{a}{1 - r}.$$

This is the *limit* to which the sum approaches as we increase the number of terms. The number of terms may be taken so great as to make the difference between the sum and $\frac{a}{1-r}$, as small as we please, but the difference will only become *zero* when the number of terms is *infinite*.

EXAMPLES.

1. Find the sum of

$$1 : \frac{1}{3} : \frac{1}{9} : \frac{1}{27} : \frac{1}{81} : \text{\&c., to infinity.}$$

We have,

$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

2. Again, take the progression

$$1 : \frac{1}{2} : \frac{1}{4} : \frac{1}{8} : \frac{1}{16} : \frac{1}{32} : \text{\&c.} \ldots \text{to infinity.}$$

We have

$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2.$$

What is the error, in each example, for $n = 4$, $n = 5$, $n = 6$?

IV. IDENTICAL EQUATIONS.

192. An **Identical Equation** is one that is true for all possible values that may be given to the unknown quantities that enter it. Thus,

$$ax^2 + bx + c = ax^2 + bx + c, \quad \text{and} \quad \frac{a^2 - x^2}{a - x} = a + x,$$

are identical equations.

193. Every identical equation containing but one unknown quantity can be reduced to the form,

$$a + bx + cx^2 + \text{\&c.} = a' + b'x + c'x^2 + \text{\&c.} \quad . \quad . \quad (1)$$

Or, by transposing, it may be reduced to the form

$$(a - a') + (b - b')x + (c - c')x^2 + \&c. = 0. \quad . \quad (2)$$

Or, denoting the coefficients of the different powers of x by single letters, it may be placed under the form

$$A + Bx + Cx^2 + Dx^3 + \&c. = 0 \quad . \quad . \quad . \quad . \quad (3)$$

In this equation x may have any value whatever; hence, it is *indeterminate.* The coefficients of the different powers of x are, by hypothesis, independent of x; and being coefficients of indeterminate quantities, they have received the name of *indeterminate coefficients.*

194. Since the values of A, B, C, D, etc., in equation (3), Article **193**, are independent of x, they will not be affected by any supposition we may make on the value of that quantity. If we make $x = 0$, in that equation, all the terms containing x reduce to 0, and we find

$$A = 0.$$

Substituting this value of A in equation (3), and factoring, it becomes,

$$x(B + Cx + Dx^2 + \&c.) = 0 \quad . \quad . \quad . \quad . \quad (4)$$

which may be satisfied by placing $x = 0$, or by placing

$$B + Cx + Dx^2 + \&c. = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The first supposition gives a common equation, satisfied only for $x = 0$. Hence, equation (4) can only be an identical equation under a supposition which makes equation (5) an identical equation.

If we make $x = 0$ in equation (5), all the terms containing x will reduce to 0, and we find

$$B = 0.$$

Substituting this value of B in equation (5), and factoring, we get

$$x(\mathrm{C} + \mathrm{D}x + \text{\&c.}) = 0 \quad . \; . \; . \; . \; . \; . \quad (6)$$

In the same manner as before, we may show that $\mathrm{C} = 0$, and so we may prove, in succession, that each of the coefficients D, E, etc., is separately equal to 0; hence,

In an identical equation containing but one unknown quantity, one member of which is equal to 0, the coefficients of the different powers of the unknown quantity are separately equal to 0.

195. Since A, B, C, etc., are separately equal to 0 in equation (3), Art. **193**, it follows that their equals in equation (2) are also equal to 0; that is,

$$a - a' = 0, \quad b - b' = 0, \quad c - c' = 0, \quad \text{etc.}$$

Hence,

$$a = a', \quad b = b', \quad c = c', \quad \text{etc.}$$

Comparing their results with equation (1), Article **153**, we see that,

In an identical equation containing but one unknown quantity, the coefficients of the like powers of that quantity in the two members are separately equal to each other.

196. We may extend the principles just deduced to identical equations containing any number of unknown quantities.

For, let us assume that the equation

$$a + bx + b'y + b''z + \text{\&c.} + cx^2 + c'y^2 + c''z^2 + \text{\&c.} + dx^3$$
$$+ d'y^3 + \text{\&c.} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

is satisfied by any values that may be assigned to x, y, z, &c. If we make all the unknown quantities, except x equal to 0, ation (1) will reduce to

$$a + bx + cx^2 + dx^3 + \text{\&c.} = 0;$$

whence, from the principle of Article **194**,

$$a = 0, \quad b = 0, \quad c = 0, \quad d = 0, \quad \&c.$$

If we make all the quantities except y equal to 0, equation (1) reduces to

$$a + b'y + c'y^2 + d'y^3 + \&c. = 0;$$

whence, as before,

$$a = 0, \quad b' = 0, \quad c' = 0, \quad d' = 0, \quad \&c.;$$

and similarly we have

$$b'' = 0, \quad c'' = 0, \quad \&c.$$

Hence, in an identical equation containing several unknown quantities, one member of which is 0, the coefficients of the different powers of the unknown quantities are separately equal to 0.

This principle, which embraces both the results of the last article, is called the *principle of indeterminate coefficients.*

V. INDETERMINATE COEFFICIENTS.

197. The principle of Indeterminate Coefficients is used in developing algebraic expressions into series.

For example, let it be required to develop

$$\frac{a}{a' + b'x},$$

into a series arranged according to the ascending powers of x.

Let us assume a development of the proposed form,

$$\frac{a}{a' + b'x} = P + Qx + Rx^2 + Sx^3 + \&c. \quad . \quad . \quad . \quad (1)$$

in which P, Q, R, etc., are independent of x, and dependent on a, a', and b', for their values. It is required to find such values for P, Q, R, etc., as will make the development true for all values of x.

Clearing of fractions, and transposing all the terms to the first member, we have

$$\begin{array}{r|r|r|l} Pa' & +\,Qa' & x + Ra' & x^2 + \&c. = 0. \\ -\,a & +\,Pb' & +\,Qb' & \end{array}$$

Since this equation is true for all values of x, it is identical, and from the principle of Art. **194**, we have

$$Pa' - a = 0, \quad Qa' + Pb' = 0, \quad Ra' + Qb' = 0, \quad \&c., \&c.;$$

whence,

$$P = \frac{a}{a'}, \quad Q = -\frac{Pb'}{a'} = -\frac{ab'}{a'^2}, \quad R = -\frac{Qb'}{a'} = \frac{ab'^2}{a'^3}, \ \&c., \&c.$$

Substituting these values of P, Q, R, etc., in equation (1), it becomes

$$\frac{a}{a' + bx} = \frac{a}{a'} - \frac{ab'}{a'^2}x + \frac{ab'^2}{a'^3}x^2 - \frac{ab'^3}{a'^4}x^3 + \&c. \quad . \quad . \quad (2)$$

Since we may operate in the same manner on any like expression, we have for developing an algebraic expression into a series, the following

RULE.

I. Place the given expression equal to a series of the form P + Qx + Rx^2 + *&c., clear the resulting equation of fractions, and transpose all of the terms into the first member:*

II. Place the coefficients of the different powers of the unknown quantity separately equal to 0, and from these equations find the values of P, Q, R, &c.

III. Having found these values, substitute them for P, Q, R, &c., in the assumed series, and the result will be the development required.

EXAMPLES.

1. Develop $\dfrac{a}{a-x}$ into a series.

Ans. $1+\dfrac{x}{a}+\dfrac{x^2}{a^2}+\dfrac{x^3}{a^3}+$ &c.

2. Develop $\dfrac{1}{(a-x)^2}$ into a series.

An. $\dfrac{1}{a^2}+\dfrac{2x}{a^3}+\dfrac{3x^2}{a^4}+\dfrac{4x^3}{a^5}+$ &c.

3. Develop $\dfrac{1+2x}{1-3x}$ into a series.

Ans. $1+5x+15x^2+45x^3+135x^4+$ &c.

4. Develop $\dfrac{1-x}{1+x+x^2}$ into a series.

Ans. $1-2x+x^2+x^3-2x^4+x^5+$ &c.

5. Develop $\dfrac{1-x}{1+x}$ into a series.

Ans. $1-2x+2x^2-2x^3+2x^4-2x^4+$ &c.

6. Develop $\dfrac{1}{1-2x+x^2}$ into a series.

Ans. $1+2x+3x^2+4x^3+5x^4+$ &c.

7. Develop $\dfrac{1}{1+2x+x^2}$ into a series.

Ans. $1-2x+3x^2-4x^3+5x^4-$ &c.

8. Develop $\dfrac{1-x}{1+2x+x^2}$ into a series.

Ans. $1-3x+5x^2-7x^3+9x^4-$ &c.

198. We have supposed the series to be arranged according to the ascending powers of the unknown quantity, commencing with the 0 power, but some expressions cannot be developed according to this law. In such cases, the application of the rule gives rise to some absurdity.

For example, if we apply the rule to develop $\frac{1}{3x-x^2}$, we have,

$$\frac{1}{3x-x^2} = \mathrm{P} + \mathrm{Q}x + \mathrm{R}x^2 + \&c. \quad . \quad . \quad . \quad . \quad (1)$$

Clearing of fractions, and transposing,

$$-1 + 3\mathrm{P}x + \left.\begin{array}{r}3\mathrm{Q} \\ -\mathrm{P}\end{array}\right| x^2 + \&c. = 0;$$

Whence, by the rule,

$$-1 = 0, \quad 3\mathrm{P} = 0, \quad 3\mathrm{Q} - \mathrm{P} = 0, \quad \&c.$$

Now, the first equation is absurd, since -1 cannot equal 0. Hence, we conclude that the expression cannot be developed according to the ascending powers of x, beginning at x^0.

We may, however, write the expression under the form $\frac{1}{x} \times \frac{1}{3-x}$, and by the application of the rule develop the factor $\frac{1}{3-x}$, which gives

$$\frac{1}{3-x} = \frac{1}{3} + \frac{1}{9}x + \frac{1}{27}x^2 + \frac{1}{81}x^3 + \&c.;$$

hence, by substitution,

$$\frac{1}{3x-x^2} = \frac{1}{3x} + \frac{1}{9}x^0 + \frac{1}{27}x + \frac{1}{81}x^2 + \&c.$$

Since $\frac{1}{3x}$ is equal to $\frac{1}{3}x^{-1}$ we see that the *true* development contains a term with a negative exponent, and the supposition made in equation (1) ought to fail.

The rule will also fail if the numerator of the fraction to be developed is of a higher degree, with respect to the unknown quantity, than the denominator. In such cases we reduce the given fraction to a mixed quantity, in which the numerator of the fractional part is of a less degree than the denominator, and then apply the rule to the fractional part, annexing the result to the entire part.

EXAMPLE.

1. Develop $\dfrac{1 + ax^2 - x^3}{a - x}$ into a series.

We have,

$$\frac{1 + ax^2 - x^3}{a - x} = x^2 + \frac{1}{a - x}.$$

Hence, the required series is

$$x^2 + \frac{1}{a} + \frac{x}{a^2} + \frac{x^2}{a^3} + \frac{x^3}{a^4} + \&c.$$

RECURRING SERIES.

199. The development of fractions of the form $\dfrac{a}{a' + b'x}$, etc., gives rise to a kind of series, called *recurring series.*

A **Recurring Series** is one in which any term is equal to the algebraic sum of the products obtained by multiplying one or more of the preceding terms by certain fixed quantities.

These fixed quantities, taken in order, constitute what is called the *scale of the series.*

If we examine the development

$$\frac{a}{a' + b'x} = \frac{a}{a'} - \frac{ab'}{a'^2}x + \frac{ab'^2}{a'^3}x^2 - \frac{ab'^3}{a'^4}x^3 + \&c.,$$

we see, that each term is formed by multiplying the preceding one by $-\dfrac{b'}{a'}x.$ This is called a recurring series of the *first order,* because the scale of the series contains but one term.

The expression $-\frac{b'}{a'}x$ is the *scale of the series*, and the expression $-\frac{b'}{a'}$ is called the *scale of coefficients.*

A geometrical progression is a recurring series of the first order.

200. Let it be required to develop the expression

$$\frac{a+bx}{a'+b'x+c'x^2} \text{ into a series. . . . . . } \quad (1)$$

Assume

$$\frac{a+bx}{a'+b'x+c'x^2} = \mathrm{P} + \mathrm{Q}x + \mathrm{R}x^2 + \mathrm{S}x^3 + \&c.$$

Clearing of fractions, and transposing, we get

$$\begin{array}{r|r|r|r|l}
\mathrm{P}a' & +\mathrm{Q}a' & x+\mathrm{R}a' & x^2+\mathrm{S}a' & x^3+\&c.=0.\\
-a & +\mathrm{P}b' & +\mathrm{Q}b' & +\mathrm{R}b' & \\
 & -b & +\mathrm{P}c' & +\mathrm{Q}c' &
\end{array}$$

Therefore, we have,

$$\mathrm{P}a' - a = 0, \quad \text{or,} \quad \mathrm{P} = \frac{a}{a'};$$

$$\mathrm{Q}a' + \mathrm{P}b' - b = 0, \quad \text{or,} \quad \mathrm{Q} = -\frac{b'}{a'}\mathrm{P} + \frac{b}{a'};$$

$$\mathrm{R}a' + \mathrm{Q}b' + \mathrm{P}c' = 0, \quad \text{or,} \quad \mathrm{R} = -\frac{b'}{a'}\mathrm{Q} - \frac{c'}{a'}\mathrm{P};$$

$$\mathrm{S}a' + \mathrm{R}b' + \mathrm{Q}c' = 0, \quad \text{or,} \quad \mathrm{S} = -\frac{b'}{a'}\mathrm{R} - \frac{c'}{a'}\mathrm{Q};$$

&c., &c., &c., &c.;

from which we see that, commencing at the third, each coefficient is formed by multiplying that which immediately precedes the required coefficient by $-\frac{b'}{a'}$, that which precedes it two terms by $-\frac{c'}{a'}$, and taking the algebraic sum the products. Hence,

$$\left(-\frac{b'}{a'},\ -\frac{c'}{a'}\right)$$

is the *scale of coefficients.*

From this law of formation of the coefficients, it follows that the third term, and every succeeding one, is formed by multiplying the one that next precedes by $-\frac{b'}{a'}x$, and the second preceding one, by $-\frac{c'}{a'}x^2$, and then taking the algebraic sum of the products: hence,

$$\left(-\frac{b'}{a'}x,\ -\frac{c'}{a'}x^2\right) \quad \ldots\ldots\ldots \quad (2)$$

is the *scales of the series.*

This scale contains two terms, and the series is called a recurring series of the *second order.*

The development of the fraction

$$\frac{a + bx + cx^2}{a' + b'x + c'x^2 + d'x^3} \quad \ldots\ldots\ldots \quad (3)$$

gives rise to a recurring series of the third order, the scale of which is

$$\left(-\frac{b'}{a'}x,\ -\frac{c'}{a'}x^2,\ -\frac{d'}{a'}x^3\right) \quad \ldots\ldots \quad (4)$$

In general, the order of a series is denoted by the number of terms in its scale.

EXAMPLES.

1. Develop $\dfrac{1}{1 - x - x^2}$ into a series.

By comparison with equation (1), we see that the series is of the second order, and that its scale is

$$(+x,\ x^2).$$

Finding two terms, $1 + x$, by actual division, and then applying the scale, we have

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \&c.$$

2. Develop $\dfrac{1 + 2x}{1 - 3x + 2x^2 - x^3}$ into a series.

The series is of the third order, and its scale is

$$(3x, \quad -2x^2, \quad x^3).$$

Finding these terms, $1 + 5x + 13x^2$, by actual division, and then applying the scale, we have

$$1 + 5x + 13x^2 + 30x^3 + 69x^4 + 160x^5 + \&c.$$

VI. BINOMIAL FORMULA FOR ANY EXPONENT.

201. It has been shown (Art. **45**) that any expression of the form $z^m - y^m$, is exactly divisible by $z - y$, when m is a positive whole number, giving

$$\frac{z^m - y^m}{z - y} = z^{m-1} + z^{m-2}y + z^{m-3}y^2 + \dots + y^{m-1}.$$

The number of terms in the quotient is equal to m, and if we suppose $z = y$, each term will become z^{m-1}; hence,

$$\left(\frac{z^m - y^m}{z - y}\right)_{y=z} = mz^{m-1}.$$

The notation employed, in the first member, indicates what the quantity within the parenthesis becomes when we make $y = z$.

We now propose to show that this *form* holds true when m is either fractional or negative.

First, suppose m fractional, and equal to $\frac{p}{q}$.

Make $z^{\frac{1}{q}} = v$, whence, $z^{\frac{p}{q}} = v^p$, and $z = v^q$;

and $y^{\frac{1}{q}} = u$, whence, $y^{\frac{p}{q}} = u^p$, and $y = u^q$:

hence,

$$\frac{z^{\frac{p}{q}} - y^{\frac{p}{q}}}{z - y} = \frac{v^p - u^p}{v^q - u^q} = \frac{\dfrac{v^p - u^p}{v - u}}{\dfrac{v^q - u^q}{v - u}}$$

If we suppose $y = z$, we have $v = u$, and since p and q are positive whole numbers, we have

$$\left\{\frac{z^{\frac{p}{q}} - y^{\frac{p}{q}}}{z - y}\right\}_{y=z} = \frac{\left(\dfrac{v^p - u^p}{v - u}\right)_{v=u}}{\left(\dfrac{v^q - u^q}{v - u}\right)_{v=u}} = \frac{pv^{p-1}}{qv^{q-1}} = \frac{p}{q}v^{p-q} = \frac{p}{q}z^{\frac{p}{q}-1}.$$

Second, suppose *m* negative, and either entire or fractional.

By observing that

$$-z^{-m}y^{-m} \times (z^m - y^m) = z^{-m} - y^{-m},$$

we have,

$$\frac{z^{-m} - y^{-m}}{z - y} = -z^{-m}y^{-m} \times \frac{z^m - y^m}{z - y}.$$

If we make the supposition that $y = z$, the first factor of the second member reduces to $-z^{-2m}$, and the second factor, from the principle just demonstrated, reduces to mz^{m-1}; hence,

$$\left(\frac{z^{-m} - y^{-m}}{z - y}\right)_{y=z} = -mz^{-m-1}.$$

We conclude, therefore, that the *form is general.*

202. By the aid of the principle demonstrated in the last article, we are able to deduce a formula for the development of

$$(x + a)^m,$$

when the exponent *m is positive or negative, entire or fractional.*

Let us assume the equation,

$$(1+z)^{m} = P + Qz + Rz^{3} + Sz^{3} + \&c. \quad . \quad . \quad . \quad (1)$$

in which P, Q, R, &c., are independent of z, but depend on m for their values. It is required to find such values for them as will make the assumed development true for every possible value of z.

If, in equation (1) we make $z = 0$, we have

$$P = 1.$$

Substituting this value for P, equation (1) becomes,

$$(1+z)^{m} = 1 + Qz + Rz^{2} + Sz^{3} + \&c. \quad . \quad . \quad . \quad (2)$$

Equation (2) being true for all values of z, let us make $z = y$; whence,

$$(1+y)^{m} = 1 + Qy + Ry^{2} + Sy^{3} + \&c. \quad . \quad . \quad (3)$$

Subtracting equation (3) from (2), member from member, and dividing the first member by $(1+z) - (1+y)$, and the second member by its equal $z - y$, we have,

$$\frac{(1+z)^{m} - (1+y)^{m}}{(1+z) - (1+y)} = Q\frac{z-y}{z-y} + R\frac{z^{2}-y^{2}}{z-y} + S\frac{z^{3}-y^{3}}{z-y} + \&c. \, . \, . \quad (4)$$

If, now, we make $1 + z = 1 + y$, whence $z = y$, the first member of equation (4), from previous principles, becomes $m(1+z)^{m-1}$, and the quotients in the second member become respectively,

$$\left(\frac{z-y}{z-y}\right)_{y=z} = 1, \quad \left(\frac{z^{2}-y^{2}}{z-y}\right)_{y=z} = 2z, \quad \left(\frac{z^{3}-y^{3}}{z-y}\right)_{y=z} = 3z^{2}, \quad \&c., \&c.$$

Substituting these results in equation (4), we have,

$$m(1+z)^{m-1} = Q + 2Rz + 3Sz^{2} + 4Tz^{3} + \&c. \quad . \quad . \quad (5)$$

Multiplying both members of equation (5) by $(1+z)$, we d,

$$m(1+z)^m = \left.\begin{array}{r} Q+2R \\ +\ Q \end{array}\right| z + \left.\begin{array}{r} 3S \\ +2R \end{array}\right| z^2 + \left.\begin{array}{r} 4T \\ +3S \end{array}\right| z^3 + \&c. \quad . \quad . \quad . \quad (6)$$

If we multiply both members of equation (2) by m, we have

$$m(1+z)^m = m + mQz + mRz^2 + mSz^3 + mTz^4 + \&c. \; . \; . \; . \; (7)$$

The second members of equations (6) and (7) are equal to each other, since the first members are the same; hence, we have the equation,

$$m + mQz + mRz^2 + mSz^3 + \&c. = \left.\begin{array}{r} Q+2R \\ +\ Q \end{array}\right| z + \left.\begin{array}{r} 3S \\ +2R \end{array}\right| z^2 + \left.\begin{array}{r} 4T \\ +3S \end{array}\right| z^3 + \&c. \; (8)$$

This equation being identical, we have (Art. **195**),

$$Q = m, \qquad \text{or,} \qquad Q = \frac{m}{1}.$$

$$2R + Q = mQ, \qquad \text{or,} \qquad R = \frac{m(m-1)}{1 \, . \, 2};$$

$$3S + 2R = mR, \qquad \text{or,} \qquad S = \frac{m(m-1)(m-2)}{1 \, . \, 2 \, . \, 3};$$

$$4T + 3S = mS, \qquad \text{or,} \qquad T = \frac{m(m-1)(m-2)m-3)}{1 \, . \, 2 \, . \, 3 \, . \, 4}.$$

&c., &c., &c.

Substituting these values in equation (2), we obtain

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{1 \, . \, 2} z^2 + \frac{m(m-1)(m-2)}{1 \, . \, 2 \, . \, 3} z^3$$

$$+ \frac{m(m-1)(m-2)(m-3)}{1 \, . \, 2 \quad 3 \, . \, 4} z^4 + \&c. \quad . \quad . \quad . \quad (9)$$

If now, in the last equation, we write $\frac{a}{x}$ for z, and then multiply both members by x^m, we shall have,

$$(x+a)^m = x^m + max^{m-1} + \frac{m(m-1)}{1\,.\,2}a^2x^{m-2} + \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}a^3x^{m-3}$$
$$+ \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Since this formula is identical with that deduced in **Art. 141**, we infer that the form of the development of $(x+a)^m$ is the same, whether m is *positive or negative, entire or fractional.*

The number of terms in the development, when m is either fractional or negative, is infinite.

EXAMPLES.

1. Develop $(1+x)^{\frac{1}{2}}$ to five terms.

If in (10), we make $x=1$, $a=x$, and $m=\frac{1}{2}$, we have,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \&c.$$

2. Develop $(1-x)^{-\frac{1}{2}}$ to five terms.

Ans. $1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 + \&c.$

203. If in the formula $(x+a)^m =$

$$x^m\left(1 + m.\frac{a}{x} + m.\frac{m-1}{2}.\frac{a^2}{x^2} + m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{a^3}{x^3} + \ .\ .\ \right)$$

we make $m=\frac{1}{n}$, it becomes $(x+a)^{\frac{1}{n}}$ or $\sqrt[n]{x+a} =$

$$x^{\frac{1}{n}}\left(1 + \frac{1}{n}.\frac{a}{x} + \frac{1}{n}.\frac{\frac{1}{n}-1}{2}.\frac{a^2}{x^2} + \frac{1}{n}.\frac{\frac{1}{n}-1}{2}.\frac{\frac{1}{n}-2}{3}.\frac{a^3}{x^3} + \ .\ .\ .\ \right)$$

or, reducing, $\sqrt[n]{x+a} =$

$$x^{\frac{1}{n}}\left(1 + \frac{1}{n}.\frac{a}{x} - \frac{1}{n}.\frac{n-1}{2n}.\frac{a^2}{x^2} + \frac{1}{n}.\frac{n-1}{2n}.\frac{2n-1}{3n}.\frac{a^3}{x^3} - \ .\ .\ \right)$$

The fifth term, within the parenthesis, can be found by multiplying the fourth by $\frac{3n-1}{4n}$ and by $\frac{a}{x}$, then changing the sign of the result, and so on.

204. Approximate Roots.—The formula just deduced is used to find an approximate root of a number. Let it be required to find the cube root of 31.

The greatest perfect cube in 31 is 27. Let $x = 27$ and $a = 4$: making these substitutions in the formula, and putting 3 in the place of n, it becomes

$$\sqrt[3]{31} = 3\left(1 + \frac{1}{3} \cdot \frac{4}{27} \cdot -\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{16}{729} + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{5}{9} \cdot \frac{64}{19683} \right.$$
$$\left. - \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{5}{9} \cdot \frac{2}{3} \cdot \frac{256}{531441} + \&c.\right)$$

or, by reducing,

$$\sqrt[3]{31} = 3 + \frac{4}{27} - \frac{16}{2187} + \frac{320}{531441} - \frac{2560}{43046721} + \&c.$$

Whence, $\sqrt[3]{31} = 3\,.\,14138$, which, as we shall show presently, is exact to within less than .00001.

We may, in like manner, treat all similar cases: hence, for extracting any root, approximatively, by the binomial formula, we have the following

RULE.

Find the perfect power of the degree indicated, which is nearest to the given number, and place this in the formula for x. Subtract this power from the given number, and substitute the difference, which may be negative, in the formula for a. Perform the operations indicated, and the result will be the required root.

EXAMPLES.

1. $\sqrt[3]{28} = 27^{\frac{1}{3}}\left(1 + \frac{1}{27}\right)^{\frac{1}{3}} = 3.0366.$

2. $\sqrt[5]{30} = (32 - 2)^{\frac{1}{5}} = 32^{\frac{1}{5}}\left(1 - \frac{1}{16}\right)^{\frac{1}{5}} = 1.9744.$

3. $\sqrt[5]{39} = (32 + 7)^{\frac{1}{5}} = 32^{\frac{1}{5}}\left(1 + \frac{7}{32}\right)^{\frac{1}{5}} = 2.0807.$

4. $\sqrt[7]{108} = (128 - 20)^{\frac{1}{7}} = 128^{\frac{1}{7}}\left(1 - \frac{5}{32}\right)^{\frac{1}{7}} = 1.95204.$

Series.

205. A **Decreasing Series** is one in which each term is numerically less than the preceding one.

An **Increasing Series** is one in which each term is greater than the preceding one.

206. A **Converging Series** is one in which the greater the number of terms taken, the nearer will their sum approximate to a fixed value, called the *sum of the series.* When the terms of a decreasing series are *alternately positive and negative,* it will be converging, and we can determine the degree of approximation attained by taking the sum of a limited number of terms for the true sum of the series.

For, let $a - b + c - d + e - f + \ldots$, &c., be a decreasing series, $b, c, d, \ldots$ being positive quantities, and let x denote the true sum of this series. Then, if n denote the number of terms taken, the value of x will be found between the sums of n and of $n + 1$ terms.

For, take any two consecutive sums,

$$a - b + c - d + e - f, \text{ and } a - b + c - d + e - f + g.$$

In the first, the terms which follow $-f$, are,

$$+g-h,\quad +k-i+\ldots;$$

but, since the series is decreasing, the terms $g-h$, $k-l$, ... &c., are positive; therefore, in order to obtain the complete value of x, a positive number must be added to the sum $a-b+c-d+e-f$. Hence, we have

$$a-b+c-d+e-f<x;$$

therefore, *x is comprehended between the first n and the first $n+1$ terms.*

But the difference between these two sums is equal to g; and since x is comprised between them, g must be greater than the difference between x and either of them; hence, *the error committed by taking the sum of n terms, $a-b+c-d+e-f$, of the series, for the sum of the series, is numerically less than the following term.*

207. Applications.—The binomial formula is also employed to develop reciprocals and binomial radicals into series.

EXAMPLES.

1. To develop the expression $\frac{1}{1-z}$, we have,

$$\frac{1}{1-z}=(1-z)^{-1}.$$

Making $m=-1$, $x=1$, and $a=-z$, in the formula, we have

$$(1-z)^{-1}=1-1\,.\,(-z)-1\,.\,\frac{-1-1}{2}\,.\,(-z)^2$$

$$-1\,.\,\frac{-1-1}{2}\,.\,\frac{-1-2}{3}\,.\,(-z)^3-\ldots$$

or, performing the operations indicated, we find for the development,

$$\frac{1}{1-z} = (1-z)^{-1} = 1 + z + z^2 + z^3 + z^4 + \&c.$$

We might have obtained this result by applying the rule for division.

2. Develop the expression, $\frac{2}{(1-z)^3}$, or, $2(1-z)^{-3}$.

Ans. $2(1 + 3z + 6z^2 + 10z^3 + 15z^4 + \&c.)$.

3. To develop the expression $\sqrt[3]{2z - z^2}$ we place it under the form $\sqrt[3]{2z} \times \left(1 - \frac{z}{2}\right)^{\frac{1}{3}}$. By the application of the formula, we find

$$\left(1 - \frac{z}{2}\right)^{\frac{1}{3}} = 1 + \frac{1}{3}\left(-\frac{z}{2}\right) + \frac{1}{3} \cdot \frac{\frac{1}{3} - 1}{2} \cdot \left(-\frac{z}{2}\right)^2 + \dots$$

$$= 1 - \frac{1}{6}z - \frac{1}{36}z^2 - \frac{5}{648}z^3 - \dots;$$

hence,

$$\sqrt[3]{2z - z^2} = \sqrt[3]{2z}\left(1 - \frac{1}{6}z - \frac{1}{36}z^2 - \frac{5}{648}z^3 - \&c.\right)$$

4. Develop $\frac{r^2}{r + x}$ into a series.

Ans. $r - x + \frac{x^2}{r} - \frac{x^3}{r^2} + \frac{x^4}{r^3}$, &c.

5. Develop the square root of $\frac{a^2 + x^2}{a^2 - x^2}$ into a series.

Ans. $1 + \frac{x^2}{a^2} + \frac{x^4}{2a^4} + \frac{x^6}{2a^6}$, &c.

6. Develop the cube root of $\frac{a^2}{(a^2 + x^2)^2}$ into a series.

Ans. $\frac{1}{a^{\frac{2}{3}}} \times \left(1 - \frac{2x^2}{3a^2} + \frac{5x^4}{9a^4} - \frac{40x^6}{81a^6}, \&c.\right)$

7. Develop $\dfrac{1}{(1-x)^3}$ to five terms.

Ans. $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \&c.$

8. Find five terms of the cube root of $a + x$.

Ans. $(a+x)^{\frac{1}{3}} = a^{\frac{1}{3}} + \dfrac{x}{3a^{\frac{2}{3}}} - \dfrac{x^2}{9a^{\frac{5}{3}}} + \dfrac{5x^3}{81a^{\frac{8}{3}}} - \dfrac{10x^4}{243a^{\frac{11}{3}}} + \&c.$

9. Develop $\dfrac{1}{c + c^{\frac{2}{3}}x^{\frac{1}{3}}}$ to five terms.

Ans. $c^{-1} - c^{-\frac{4}{3}}x^{\frac{1}{3}} + c^{-\frac{5}{3}}x^{\frac{2}{3}} - c^{-2}x + c^{-\frac{7}{3}}x^{\frac{4}{3}} + \&c.$

VII. SUMMATION OF SERIES.

208. The Summation of a Series is the operation of finding an expression for the sum of any number of terms. Many useful series may be *summed* by the aid of two auxiliary series.

Let there be a *given* series, whose terms may be derived from the expression $\dfrac{q}{n(n+p)}$, by giving to p a fixed value, and then attributing suitable values to q and n.

Let there be two auxiliary series formed from the expressions $\dfrac{q}{n}$ and $\dfrac{q}{n+p}$, so that the values of p, q, and n, shall be the same as in the corresponding terms of the first series.

It can be shown that any term of the first series is equal to $\dfrac{1}{p}$ multiplied by the excess of the corresponding term in the second series, over that in the third.

For, if we take the expression

$$\frac{1}{p}\left(\frac{q}{n} - \frac{q}{n+p}\right),$$

and perform the operations indicated, we shall get the expression,

$$\frac{q}{n(n+p)}.$$

Hence, we have,

$$\frac{q}{n(n+p)} = \frac{1}{p}\left(\frac{q}{n} - \frac{q}{n+p}\right);$$

which has to be proved.

It follows, therefore, that *the sum of any number of terms of the first series, is equal to* $\frac{1}{p}$ *multiplied by the excess of the sum of the corresponding terms in the second series, over that of the corresponding terms in the third series.*

Whenever, therefore, we can find this last difference, it is always possible to sum the given series.

EXAMPLES.

1. Required the sum of n terms of the series,

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \text{\&c.}$$

Comparing the terms of this series with the expression

$$\frac{q}{n(n+p)},$$

we see that the series may be formed by making $p = 1$, $q = 1$, and $n = 1, 2, 3, 4$, &c., in succession.

The two corresponding auxiliary series, to n terms, are

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots \frac{1}{n},$$

and

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots \frac{1}{n} + \frac{1}{n+1}.$$

The difference between the sums of n terms of the first and second auxiliary series is

$$1-\frac{1}{n+1},$$

or, if we denote the sum of n terms of the given series by S, we have,

$$S=1-\frac{1}{n+1}.$$

If the number of terms is infinite $n=\infty$ and

$$S=1.$$

2. Required the sum of n terms of the series

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\frac{1}{9.11}+\text{\&c.}$$

If we compare the terms of this series with the expression

$$\frac{q}{n(n+p)},$$

we see that $p=2$, $q=1$, and $n=1, 3, 5, 7$, &c., in succession.

The two auxiliary series, to n terms, are,

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots+\frac{1}{2n-1},$$

and

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots+\frac{1}{2n-1}+\frac{1}{2n+1};$$

hence, as before,

$$S=\frac{1}{2}\left(1-\frac{1}{2n+1}\right).$$

If $n=\infty$, we find $S=\frac{1}{2}$.

3. Required the sum of n terms of the series

$$\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+\text{\&c.}$$

Here

$$p=3, \quad q=1, \quad n=1,\ 2,\ 3,\ 4,\ \text{\&c.}$$

The two auxiliary series, to n terms, are,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots+\frac{1}{n},$$

$$\frac{1}{4}+\frac{1}{5}\ldots\ldots+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3};$$

hence,

$$S=\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right).$$

If

$$n=\infty,\ S=\frac{11}{18}.$$

4. Required the sum of the series

$$\frac{4}{1.5}+\frac{4}{5.9}+\frac{4}{9.13}+\frac{4}{13.17}+\frac{4}{17.21}+\text{\&c.}$$

Ans. 1.

5. Find the sum of n terms of the series,

$$\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}+\frac{6}{11.13}-\text{\&c.}\ldots$$

Here

$$p=2, \quad q=2, \quad -3, \quad +4, \quad -5, \quad +6, \quad \text{\&c.}$$

$$n=3,\ 5,\ 7,\ 9,\ 11,\ \text{\&c.}$$

If n is *even*, the upper sign is used, and the quantity in the last parenthesis becomes $+1$, in which case

$$S=\frac{1}{2}\left(\frac{2}{3}+\frac{n+1}{2n+3}\right)-\frac{1}{2}=\frac{1}{2}\left(-\frac{1}{3}+\frac{n+1}{2n+3}\right).$$

If n is *odd*, the lower sign is used, and the quantity in the last parenthesis becomes 0, in which case

$$S=\frac{1}{2}\left(\frac{2}{3}-\frac{n+1}{2n+3}\right).$$

If in either formula we make

$$n=\infty,\quad \frac{n+1}{2n+3}=\frac{1+\frac{1}{n}}{2+\frac{3}{n}}\ \text{ becomes }\ \frac{1}{2},\ \text{ and }\ S=\frac{1}{12}.$$

6. Find the sum of n terms of the series,

$$\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\frac{1}{4.6},\ \&c.$$

Here,

$$p=2,\ q=1,\ -1,\ +1,\ -1,\ +1,\ -1,\ \&c.$$
$$n=1,\ 2,\ 3,\ 4,\ \&c.;$$

whence,

$$S=\frac{1}{2}\left(\frac{1}{2}\mp\frac{1}{n+1}\pm\frac{1}{n+2}\right).$$

If $n=\infty$, we find $S=\frac{1}{4}$.

Method by Differences.

209. Let $a,\ b,\ c,\ d\ \ldots\ldots$ &c., represent the successive terms of a series formed according to any law; then if each term be subtracted from the succeeding one, the several remainders will form a new series called the *first order of differences*. If we subtract each term of this series from the succeeding one, we shall form another series called the *second order of differences*, and so on, as exhibited in the annexed table.

$$\begin{array}{llllll}
a, & b, & c & d, & e, & \\
 & b-a, & c-b, & d-c, & e-d,\ \&c., & \text{1st.} \\
 & & c-2b+a, & d-2c+b, & e-2d+c,\ \&c., & \text{2d.} \\
 & & & d-3c+3b-a, & e-3d+3c-b,\ \&c., & \text{3d.} \\
 & & & & e-4d+6c-4b+a,\ \&c., & \text{4th.}
\end{array}$$

If, now, we designate the first terms of the first, second, third, &c., orders of differences, by d_1, d_2, d_3, d_4, &c., we shall have,

$$\begin{array}{ll} d_1 = b - a, & \text{whence, } b = a + d_1, \\ d_2 = c - 2b + a, & \text{whence, } c = a + 2d_1 + d_2, \\ d_3 = d - 3c + 3b - a, & \text{whence, } d = a + 3d_1 + 3d_2 + d_3, \\ d_4 = e - 4d + 6c - 4b + a, & \text{whence, } e = a + 4d_1 + 6d_2 + 4d_3 + d_4, \\ \quad \&c. \qquad \&c. & \qquad\qquad \&c. \qquad \&c. \end{array}$$

And if we designate the term of the series which has n terms before it, by T, we shall find, by a continuation of the above process,

$$T = a + nd_1 + \frac{n\,.\,(n-1)}{1\,.\,2}d_2 + \frac{n\,.\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}d_3$$

$$+ \frac{n\,.\,(n-1)\,(n-2)\,(n-3)}{1\,.\,2\,.\,3\,.\,4}d_4 + \&c. \quad . \quad . \quad . \quad . \quad (1)$$

This formula enables us to find the $(n+1)^{th}$ term of a series when we know the first terms of the successive orders of differences.

210. To find an expression for the sum of n terms of the series, a, b, c, &c, let us take the series

$$0, \; a, \; a+b, \; a+b+c, \; a+b+c+d, \; \&c. \; . \; . \; (2)$$

The first order of differences is evidently

$$a, \; b, \; c, \; d, \ldots\ldots \&c. \; . \; . \; . \; . \; . \; . \; (3)$$

Now, it is obvious that the sum of n terms of the series (3) is equal to the $(n+1)^{th}$ term of the series (2).

But the first term of the first order of differences in series (2) is a; the first term of the second order of differences is the same as d_1 in equation (1). The first term of the third order of differences is equal to d_2, and so on.

Hence, making these changes in formula (1), and denoting the sum of n terms by S, we have,

$$S = na + \frac{n(n-1)}{1\,.\,2}d_1 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}d_2 + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}d_3 + \&c. \quad \ldots \ldots \quad (4)$$

When all of the terms of any order of differences become equal, the terms of all succeeding orders of differences are 0, and formulas (1) and (4) give exact results. When there are no orders of differences, whose terms become equal, then the formulas do not give exact results, but approximations, more or less exact according to the number of terms used.

EXAMPLES.

1. Find the sum of n terms of the series 1 . 2, 2 . 3, 3 . 4, 4 . 5, &c.

Series, 1 . 2, 2 . 3, 3 . 4, 4 . 5, 5 . 6, &c.
1st order of differences, 4, 6, 8, 10, &c.
2d order of differences, 2, 2, 2, &c.
3d order of differences, 0, 0.

Hence, we have, $a = 2$, $d_1 = 4$, $d_2 = 2$, d_3, d_4, &c., equal to 0.

Substituting these values for a, d_1, d_2, &c., in formula (4), we find,

$$S = 2n + \frac{n(n-1)}{1\,.\,2} \times 4 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} \times 2;$$

whence,

$$S = \frac{n(n+1)(n+2)}{3}$$

2. Find the sum of n terms of the series 1 . 2 . 3, 2 . 3 . 4, 3 . 4 . 5, 4 . 5 . 6, &c.

1st order of differences, 18, 36, 60, 90, 126, &c.
2d order of differences, 18, 24, 30, 36, &c.
3d order of differences, 6, 6, 6, &c.
4th order of differences, 0, 0, &c.

We find $a = 6$, $d_1 = 18$, $d_2 = 18$, $d_3 = 6$, $d_4 = 0$, &c.

Substituting in equation (4), and reducing, we find,

$$S = \frac{n(n+1)(n+2)(n+3)}{4}.$$

3. Find the sum of n terms of the series 1, $1+2$, $1+2+3$, $1+2+3+4$, &c.

We find, $a=1$, $d_1=2$, $d_2=1$, $d_3=0$, $d_4=0$, &c.;

$$\text{hence, } S = n + \frac{n(n-1)}{1\,.\,2}.2 + \frac{n(n-1)(n-2)}{1.2.3} = \frac{n^3+3n^2+2n}{1.2.3};$$

$$\text{or, reducing,} \qquad S = \frac{n(n+1)(n+2)}{1.2.3}.$$

4. Find the sum of n terms of the series 1^2, 2^2, 3^2, 4^2, 5^2, &c.

We find, $a=1$, $d_1=3$, $d_2=2$, $d_3=0$, $d_4=0$, &c., &c.

Substituting these values in formula (4), and reducing, we find,

$$S = \frac{n(n+1)(2n+1)}{1.2.3}.$$

5. Find the sum of n terms of the series,

$$1(m+1), \quad 2(m+2), \quad 3(m+3), \quad 4(m+4), \text{ \&c.}$$

We find, $a=m+1$, $d_1=m+3$, $d_2=2$, $d_3=0$, &c.;

$$\text{whence, } S = n(m+1) + \frac{n\,.\,(-1)}{1\,.\,2}(m+3) + \frac{n\,.\,(n-1)(n-2)}{1\,.\,2\,.\,3} \times 2.$$

$$\text{or,} \qquad S = \frac{n\,.\,(n+1)\,.\,(1+2n+3m)}{1\quad.\quad2\quad.\quad3}$$

Of Piling Balls.

The last three formulas deduced, are of practical application in determining the number of balls in different shaped piles.

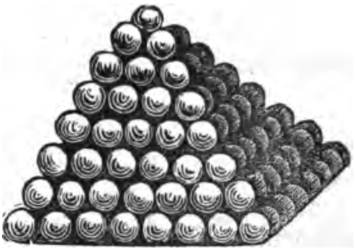
First, in the Triangular Pile.

211. A triangular pile is formed of successive triangular layers, such that the number of shot in each side of the layers, decreases continuously by 1 to the single shot at the top. The number of balls in a complete triangular pile is evidently equal to the sum of the series 1, $1+2$, $1+2+3$, $1+2+3+4$, &c., to $1+2+\ldots+n$, n denoting the number of balls on one side of the base.

But from example 3d, last article, we find the sum of n terms of the series,

$$S = \frac{n(n+1).(n+2)}{1 \;.\; 2 \;.\; 3} \quad . \; . \; . \; . \; . \; : \quad (1)$$

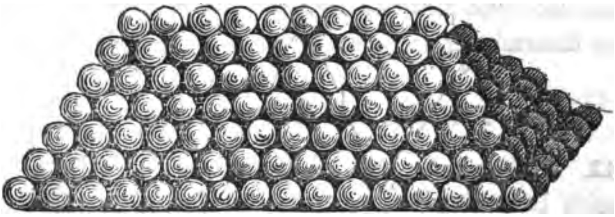
Second, in the Square Pile.



212. The square pile is formed, as shown in the figure. The number of balls in the top layer is 1; the number in the second layer is denoted by 2^2; in the next, by 3^2, and so on. Hence, the number of balls in a pile of n layers is equal to the sum of the series, 1^2, 2^2, 3^2, &c., n^2, which we see, from example 4th of the last article, is

$$S = \frac{n.(n+1).(2n+1)}{1 \;.\; 2 \;.\; 3} \quad . \; . \; . \; . \; . \quad (2)$$

Third, in the Oblong Pile.



213. The complete oblong pile has $(m+1)$ balls in the upper layer, $2\,(m+2)$ in the next layer, $3\,(m+3)$ in the third, and so on: hence, the number of balls in the complete pile, is given by the formula deduced in example 5th of the preceding article,

$$S = \frac{n\,.\,(\ +1)\,.\,(1+2n+3m)}{1\,.\,2\,.\,3} \quad \ldots\ldots \quad (3)$$

214. If any of these piles is incomplete, compute the number of balls that it would contain if complete, and the number that would be required to complete it; the excess of the former over the latter will be the number of balls in the pile.

The formulas (1), (2), and (3) may be written,

triangular, $$S = \frac{1}{3}\,.\,\frac{n(n+1)}{2}\,(n+1+1) \quad \ldots\ldots \quad (1)$$

square, $$S = \frac{1}{3}\,.\,\frac{n(n+1)}{2}\,(n+n+1) \quad \ldots\ldots \quad (2)$$

rectangular, $$S = \frac{1}{3}\,.\,\frac{n(n+1)}{2}\Big((n+m)+(n+m)+(m+1)\Big)\,.\,(3)$$

Now, since $\frac{n(n+1)}{2}$ is the number of balls in the triangular face of each pile, and the next factor, the number balls in the longest line of the base, plus the number in the side of the base opposite, plus the parallel top row, we have the following

RULE.

Add to the number of balls in the longest line of the base the number in the parallel side opposite, and also the number in the top parallel row; then multiply this sum by one-third the number in the triangular face; the product will be the number of balls in the pile.

EXAMPLES.

1. How many balls in a triangular pile of 15 courses?
Ans. 680.

2. How many balls in a square pile of 14 courses? and how many will remain after 5 courses are removed?
Ans. 1015 and 960.

3. In an oblong pile, the length and breadth at bottom are respectively 60 and 30: how many balls does it contain?
Ans. 23405.

4. In an incomplete oblong pile, the length and breadth at bottom are respectively 46 and 20, and the length and breadth at top 35 and 9: how many balls does it contain?
Ans. 7190.

5. How many balls in an incomplete triangular pile, the number of balls in each side of the lower course being 20, and in each side of the upper, 10?

6. How many balls in an incomplete square pile, the number in each side of the lower course being 15, and in each side of the upper course, 6?

7. How many balls in an incomplete oblong pile, the numbers in the lower courses being 92 and 40; and the numbers in the corresponding top courses being 70 and 18?

CHAPTER IX.

LOGARITHMS, INTEREST, COMBINATIONS, AND PROBABILITIES.

I. OF LOGARITHMS.

215. The **Logarithm of a Number** is the exponent of the power to which it is necessary to raise a fixed number to produce the given number. The fixed number is called the *base of the system.*

216. If we denote any positive number, except 1, by a, any positive number whatever by n, and the exponent of the power to which a must be raised to produce N, by x, we shall have the exponential equation,

$$a^x = \text{N} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

In this equation, a is the base of a system of logarithms, N is any positive number, and x is its logarithm.

It is plain that a cannot be negative, inasmuch as all the odd powers of a negative quantity are negative, and all its even fractional powers are imaginary; neither can it be equal to 1, for all the powers of 1, are 1.

217. If, for a given value of a, we suppose N in equation (1) to assume, in succession, every possible value from 0 to ∞, the corresponding values of x, taken together, will constitute what is called a *System of Logarithms.*

Since a may have any one of an infinite number of values, see that there may be an infinite number of systems of logarithms. Of these, only two are in general use, viz.: that whose base is 10, called the *Common System*, and that whose base is 2.718281828 . ., called the *Napierian System.*

The Napierian System is so called from its inventor, Baron Napier, a Scotch mathematician.

218. If we raise 10 to its successive powers, we may, from the definition of a logarithm, write the following

TABLE.

$10^0 = 1$; $\therefore$ $\log 1 = 0$.
$10^1 = 10$; $\therefore$ $\log 10 = 1$.
$10^2 = 100$; $\therefore$ $\log 100 = 2$.
$10^3 = 1000$; $\therefore$ $\log 1000 = 3$.
&c. &c.

We see, that in the common system, the logarithm of any number between 1 and 10 is found between 0 and 1; the logarithm of any number between 10 and 100 is between 1 and 2; the logarithm of any number between 100 and 1000 is between 2 and 3; and so on.

The logarithm of any whole number that is not a perfect power of the base, will be equal to a whole number, *plus* a part, less than 1, and not commensurable with it. This part is generally expressed decimally. The entire part is called the *characteristic*, and the decimal part, the *mantissa.*

By examining the table, we see that if a number is expressed by a single figure, the characteristic of its logarithm is 0; if it is expressed by two figures, the characteristic of its logarithm is 1; if it is expressed by three figures, the characteristic is 2; and if it is expressed by n places of figures, the characteristic is $n - 1$.

That is, *the characteristic of the logarithm of a whole number is positive and 1 less than the number of places of figures in the number.*

If we form the powers of 10 denoted by negative exponents, we have, from the definition of a logarithm, the following

TABLE.

$$
\begin{array}{llll}
(10)^{-1} = \frac{1}{10} = .1; & \therefore & \log .1 & = -1. \\
(10)^{-2} = \frac{1}{100} = .01; & \therefore & \log .01 & = -2. \\
(10)^{-3} = \frac{1}{1000} = .001; & \therefore & \log .001 & = -3. \\
\quad \&c., \quad \&c. & & \&c., & \&c.
\end{array}
$$

Here we see that the logarithm of every number between 1 and .1 is found between 0 and -1; that is, it is equal to -1, *plus* a part less than 1. The logarithm of every number between .1 and .01 is between -1 and -2; that is, it is equal to -2, *plus* a part. The logarithm of every number between .01 and .001 is between -2 and -3, or it is equal to -3, *plus* a part, and so on.

In the first case, the characteristic is -1, in the second, -2, in the third -3, and so on.

That is, the characteristic of the logarithm of a decimal fraction, is negative and numerically one greater than the number of ciphers that immediately follow the decimal point.

The decimal part of a logarithm is always regarded as positive, and to indicate that the negative sign extends only to the characteristic, it is generally written over it; thus,

$\log 0.012 = \bar{2}.079181$, equivalent to -2 prefixed to $+.079181$.

219. **A Table of Logarithms** is a table containing a set of numbers, and their logarithms so arranged that we may, by its aid, find the logarithm corresponding to any given number, or the number corresponding to any given logarithm.

The following table shows the logarithms of all numbers, from 1 to 100.

TABLE.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	26	1.414973	51	1.707570	76	1.880814
2	0.301030	27	1.431364	52	1.716003	77	1.886491
3	0.477121	28	1.447158	53	1.724276	78	1.892095
4	0.602060	29	1.462398	54	1.732394	79	1.897627
5	0.698970	30	1.477121	55	1.740363	80	1.903090
6	0.778151	31	1.491362	56	1.748188	81	1.908485
7	0.845098	32	1.505150	57	1.755875	82	1.913814
8	0.903090	33	1.518514	58	1.763428	83	1.919078
9	0.954243	34	1.531479	59	1.770852	84	1.924279
10	1.000000	35	1.544068	60	1.778151	85	1.929419
11	1.041393	36	1.556303	61	1.785330	86	1.934498
12	1.079181	37	1.568202	62	1.792392	87	1.939519
13	1.113943	38	1.579784	63	1.799341	88	1.944483
14	1.146128	39	1.591065	64	1.806180	89	1.949390
15	1.176091	40	1.602060	65	1.812913	90	1.954243
16	1.204120	41	1.612784	66	1.819544	91	1.959041
17	1.230449	42	1.623249	67	1.826075	92	1.963788
18	1.255273	43	1.633468	68	1.832509	93	1.968483
19	1.278754	44	1.643453	69	1.838849	94	1.973128
20	1.301030	45	1.653213	70	1.845098	95	1.977724
21	1.322219	46	1.662758	71	1.851258	96	1.982271
22	1.342423	47	1.672098	72	1.857333	97	1.986772
23	1.361728	48	1.681241	73	1.863323	98	1.991226
24	1.380211	49	1.690196	74	1.869232	99	1.995635
25	1.397940	50	1.698970	75	1.875061	100	2.000000

When a number exceeds 100, the characteristic of its logarithm is not written in the table, but is found by the rule. Thus, in looking for the logarithm of 2970, in a full table of logarithms, we find opposite that number, the number 472756; this is the mantissa; and since the number 2970 contains four places of figures, its characteristic is 3. Hence,

$$\log\ 2970 = 3.472756.$$

In like manner, we find,

$$\log\ .00297 = \bar{3}.472756.$$

General Properties of Logarithms.

220. The general properties of logarithms are entirely independent of the value of the base of the system in which they are taken. In order to deduce these properties, let us assume the exponential equation,

$$a^x = N,$$

in which a is any positive number except 1, N any positive number whatever, and x its logarithm in the system whose base is a.

221. If we denote any two numbers by N' and N'', and their logarithms, taken in the system whose base is a, by x' and x'', we shall have, from the preceding article,

$$a^{x'} = N' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$a^{x''} = N'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we multiply equations (1) and (2) together, member by member, we get,

$$a^{x'+x''} = N' \times N'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But since a is the base of the system, we have from the definition,

$$x' + x'' = \log(N' \times N''); \text{ that is,}$$

The logarithm of the product of two numbers is equal to the sum of their logarithms.

If we divide equation (1) by equation (2), member by member, we have,

$$a^{x'-x''} = \frac{N'}{N''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But, from the definition,

$$x' - x'' = \log\left(\frac{N'}{N''}\right); \text{ that is,}$$

The logarithm of the quotient of one number by another, is equal to the logarithm of the dividend minus the logarithm of the divisor.

222. If we raise both members of equation (1) to the n^{th} power, we have,

$$a^{nx'} = N'^{n} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

But from the definition, we have,

$$nx' = \log (N'^{n}); \text{ that is,}$$

The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

223. If we extract the n^{th} root of both members of equation (1), we shall have,

$$a^{\frac{x'}{n}} = (N')^{\frac{1}{n}} = \sqrt[n]{N'} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

But from the definition,

$$\frac{x'}{n} = \log (\sqrt[n]{N'}); \text{ that is,}$$

The logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.

224. Practical Rules.—From the principles demonstrated in the four preceding articles, we deduce the following practical rules:

1°. To multiply quantities by means of their logarithms.

Find from a table, the logarithms of the given factors; take the sum of these logarithms, and look in the table for the corresponding number; this will be the product required.

Thus,

log 7	0.845098
log 8	0.903090
log 56	1.748188

hence, $7 \times 8 = 56.$

2°. To divide quantities by means of their logarithms.

Find the logarithms of the dividend and divisor, from a table; subtract the latter from the former, and look for the number corresponding to this difference; this will be the quotient required.

Thus,

$$\begin{array}{ll} \log 84 \;\ldots\ldots\ldots\ldots & 1.924279 \\ \log 21 \;\ldots\ldots\ldots\ldots & \underline{1.322219} \\ \log \;\, 4 \;\ldots\ldots\ldots\ldots & \underline{0.602060}; \end{array}$$

hence,

$$\frac{84}{21} = 4.$$

3°. To raise a number to any power.

Find from a table the logarithm of the number, and multiply it by the exponent of the required power; find the number corresponding to this product, and it will be the required power.

Thus,

$$\begin{array}{lr} \log \;\, 4 \;\ldots\ldots\ldots\ldots & 0.602060 \\ & \underline{3} \\ \log 64 \;\ldots\ldots\ldots\ldots & 1.806180; \end{array}$$

hence,

$$(4)^3 = 64.$$

4°. To extract any root of a number.

Find from a table the logarithm of the number, and divide this by the index of the root; find the number corresponding to this quotient, and it will be the root required.

Thus,

$$\begin{array}{ll} \log 64 \;\ldots\ldots\ldots\ldots & \underline{1.806180 \div 6} \\ \log \;\, 2 \;\ldots\ldots\ldots\ldots & \underline{.301030;} \end{array}$$

hence,

$$\sqrt[6]{64} = 2.$$

225. The following examples illustrate the methods of transforming and simplifying algebraic expressions by means of the principles demonstrated in the preceding articles:

EXAMPLES.

1. $\log(a \, . \, b \, . \, c \, . \, d \, . \dots) = \log a + \log b + \log c \dots .$

2. $\log\left(\frac{abc}{de}\right) = \log a + \log b + \log c - \log d - \log e.$

3. $\log(a^m \, . \, b^n \, . \, c^p \dots) = m \log a + n \log b + p \log c + \dots .$

4. $\log(a^2 - x^2) = \log(a + x) + \log(a - x).$

5. $\log \sqrt{(a^2 - x^2} = \frac{1}{2}\log(a + x) + \frac{1}{2}\log(a - x).$

6. $\log(a^3 \times \sqrt[4]{a^3}) = 3\frac{3}{4} \log a.$

Transform the following equations:

7. $y = \frac{x^4}{(a^2 - x^2)^{\frac{1}{2}}}.$

$\quad$ *Ans.* $\log y = 4 \log x - \frac{1}{2}[\log(a + x) + \log(a - x)].$

8. $y = \frac{x^3}{(1 + x)^{\frac{2}{3}}}.$ *Ans.* $\log y = 3 \log x - \frac{2}{3}\log(1 + x).$

9. $y = \frac{x^2 - 4}{x^3(x - 1)}.$

$\quad$ *Ans.* $\log y = \log(x + 2) + \log(x - 2) - 3 \log x - \log(x - 1).$

10. $x = \frac{a^{\frac{3}{2}} y^{-\frac{2}{5}}}{b^{\frac{1}{4}}}.$ *Ans.* $\log x = \frac{3}{2}\log a - \frac{2}{5}\log y - \frac{1}{4}\log b.$

11. $x = \frac{\sqrt{7} \times \sqrt[3]{3}}{\sqrt[7]{2} \times \sqrt[4]{5}}.$

$\quad$ *Ans.* $\log x = \frac{1}{2}\log 7 + \frac{1}{3}\log 3 - \frac{1}{7}\log 2 - \frac{1}{4}\log 5.$

The use of these simplifications is to facilitate the solution of equations in certain cases; also to shorten the process of finding numerical values of complex expressions, similar to those in the second member of example 11.

Discussion of the Exponential Equation.

226. Let us resume the exponential equation

$$a^x = \text{N}.$$

By discussing this equation, we discover several important properties of logarithms

1°. If we make $\text{N} = 1$, x must be equal to 0, since $a^0 = 1$; that is,

The logarithm of 1 *in any system is* 0.

2°. If we make $\text{N} = a$, x must be equal to 1, since $a^1 = a$; that is,

Whatever be the base of a system, its logarithm, taken in that system, is equal to 1.

3°. Suppose $a > 1$;

then, when $\text{N} = 1$, $x = 0$; when $\text{N} > 1$, $x > 0$; when $\text{N} < 1$, $x < 0$, or negative; that is,

In a system whose base is greater than 1, *the logarithms of all numbers greater than* 1 *are positive, those of all numbers less than* 1 *are negative.*

4°. Suppose $a < 1$;

then, when $\text{N} = 1$, $x = 0$; when $\text{N} < 1$, $x > 0$; when $\text{N} > 1$, $x < 0$, or negative; that is,

In a system whose base is less than 1, *the logarithms of all numbers greater than* 1 *are negative, and those of all numbers less than* 1 *are positive.*

5°. When $a > 1$ and $\text{N} < 1$, we have,

$$a^{-x} = \text{N}, \text{ or, } \frac{1}{a^x} = \text{N}.$$

Now, if N be diminished, the corresponding values of x will be increased; and when N becomes less than any assignable quantity, or 0, the value of x must be ∞; that is,

The logarithm of 0, *in a system whose base is greater than* 1, *is equal to* $-\infty$.

6°. When $a < 1$ and $N < 1$, we have,

$$a^x = N, \quad \text{or,} \quad \frac{1}{a^x} = \frac{1}{N};$$

in which, if N be diminished, the value of x will be increased; and finally, when $N = 0$, we shall have $x = \infty$; that is,

The logarithm of 0, *in a system whose base is less than* 1, is equal to $+\infty$.

7°. Whatever value may be given to x, the value of N must be positive; hence, there is no value of x that will make N negative; that is,

Negative numbers have no logarithms.

Logarithmic Series.

227. Let it be required to develop $\log(1 + y)$ into a series arranged according to the ascending powers of y. The series must be such, that it will reduce to 0 when $y = 0$; for, when $y = 0$, $\log(1 + y)$ becomes $\log 1$, which is equal to 0, in any system. Hence, y must enter every term.

Assume, therefore, the identical equation,

$$\log(1 + y) = My + Ny^2 + Py^3 + Qy^4 + \&c. \quad . \quad . \quad (1)$$

in which M, N, P, Q, &c., depend on a, the base of the system. It is now required to find such values of M, N, P, Q, &c., as will satisfy equation (1) for all values of y.

Since equation (1) is to be satisfied for any value of y, we may write z for y; whence,

$$\log(1+z) = Mz + Nz^2 + Pz^3 + Qz^4 + \&c. \quad . \quad . \quad (2)$$

Subtracting equation (2) from equation (1), member from member, we obtain,

$$\log(1+y) - \log(1+z) = M(y-z) + N(y^2-z^2) + P(y^3-z^3) \\ + Q(y^4-z^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The second member of (3) is divisible by $y-z$. To reduce the first member to such a form that it will also be divisible by $y-z$, we have

$$\log(1+y) - \log(1+z) = \log\left(\frac{1+y}{1+z}\right) = \log\left(1+\frac{y-z}{1+z}\right).$$

But since $\frac{y-z}{1+z}$ can be regarded as a single quantity, we may substitute it for y in equation (1), which gives,

$$\log\left(1+\frac{y-z}{1+z}\right) = M\left(\frac{y-z}{1+z}\right) + N\left(\frac{y-z}{1+z}\right)^2 + P\left(\frac{y-z}{1+z}\right)^3 + \&c.$$

Substituting this development for its equal, in the first member of equation (3), and dividing both members of the resulting equation by $(y-z)$, we have,

$$M\left(\frac{1}{1+z}\right) + N\frac{(y-z)}{(1+z)^2} + P\frac{(y-z)^2}{(1+z)^3} + \&c. = M + N(y+z) \\ + P(y^2+yz+z^2) + \&c.$$

Since this equation is true for all values of y and z, make $z = y$, and there will result

$$\frac{M}{1+y} = M + 2Ny + 3Py^2 + 4Qy^3 + 5Ry^4 + \&c.$$

Clearing of fractions, and transposing, we obtain,

$$\begin{array}{l|l|l|l|l} +M+2N & y+3P & y^2+4Q & y^3+5R & y^4+\&c. = 0; \\ -M+\ \ M & \ \ +2N & \ \ +3P & \ \ +4Q & \end{array}$$

Since this equation is identical, we have,

$$\begin{array}{lll} M - M = 0; & \text{whence,} & M = M; \\ 2N + M = 0; & \text{whence,} & N = -\dfrac{M}{2}; \\ 3P + 2N = 0; & \text{whence,} & P = -\dfrac{2N}{3} = \dfrac{M}{3}; \\ 4Q + 3P = 0; & \text{whence,} & Q = -\dfrac{3P}{4} = -\dfrac{M}{4}. \\ \quad \&c. & & \quad \&c. \end{array}$$

Substituting these values for N, P, Q, &c., in equation (1), we find for $\log(1 + y)$,

$$\log(1 + y) = My - \frac{M}{2}y^2 + \frac{M}{3}y^3 - \frac{M}{4}y^4 \,.\,.\, \&c.$$

$$= M\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} \,.\,.\, \&c.\right) \quad .\ . \quad (4)$$

which is called the *Logarithmic Series.*

The value of M depends on a, the base of the system, and the value of y depends on the number whose logarithm is sought; hence,

The expression for the logarithm of any number is composed of two factors, one dependent on the number, and the other on the base of the system in which the logarithm is taken.

The factor that depends on the base is called the *Modulus* of the system.

228. If we take the logarithm of $1 + y$ in a new system, and denote it by $l(1 + y)$, we shall have,

$$l(1 + y) = M'\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \&c.\right) \,.\,.\, (5)$$

in which M′ is the modulus of the new system.

If we suppose y to have the same value in equations (4) and (5), and divide the former by the latter, member by member, we have $\frac{\log(1+y)}{l(1+y)} = \frac{M}{M'}$; whence (Art. **182**),

$$l(1+y) : \log(1+y) :: M' : M; \text{ hence,}$$

The logarithms of the same number, taken in two different systems, are to each other as the moduli of those systems.

229. Since the modulus and base are mutually dependent on each other, we may assume a value for either of them, and from it deduce a value for the other. If we assume the modulus equal to 1, and find the base, we have the Napierian system. If we assume the base equal to 10, and find the modulus, we have the common system.

If we designate the Napierian logarithm of a number by the symbol l, and any other logarithm by the symbol *log*, we have, from the last article,

$$l(1+y) : \log(1+y) :: 1 : M;$$

whence,

$$M \times l(1+y) = \log(1+y) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence, we see that,

The Napierian logarithm of any number, multiplied by the modulus of any other system, will give the logarithm of the same number in that system.

Modulus.—The modulus of the Napierian System being 1, it is found most convenient to compare all other systems with the Napierian; and hence, the modulus of any system may be defined to be,

The number by which it is necessary to multiply the Napierian logarithm in order to obtain the logarithm of the same number in that system.

In equation (4), the second factor, in the second member, is the Napierian logarithm of the first member, taken in any system.

230. Again, $\mathrm{M} \times l(1+y) = \log(1+y)$, gives

$$l(1+y) = \frac{\log(1+y)}{\mathrm{M}}; \quad . \quad . \quad . \quad (2)$$

That is,

The logarithm of any number divided by the modulus of its system, is equal to the Napierian logarithm of the same number.

231. The logarithmic series is not sufficiently converging to make it convenient for computing logarithms. To deduce a more suitable series for this purpose, let us assume the logarithmic series, and make $\mathrm{M} = 1$, and denote the corresponding logarithm, as before, by the symbol l: we have,

$$l(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \&c. \quad . \quad . \quad . \quad . \quad (1)$$

If now, we write in equation (1), $-y$ for y, it becomes,

$$l(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \frac{y^5}{5} - \&c. \quad . \quad . \quad . \quad (2)$$

Subtracting equation (2) from (1), member from member, we have,

$$l(1+y) - l(1-y) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} + \frac{y^9}{9} + \&c.\right). \quad . \quad (3)$$

But,

$$l(1+y) - l(1-y) = l\left(\frac{1+y}{1-y}\right); \quad \text{whence,}$$

$$l\left(\frac{1+y}{1-y}\right) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} + \frac{y^9}{9} + \&c.\right) \quad . \quad . \quad (4)$$

If we make $\dfrac{1+y}{1-y} = \dfrac{z+1}{z}$, we have,

$$(1+y)z = (1-y)(z+1), \quad \text{whence,} \quad y = \frac{1}{2z+1}.$$

Substituting these values in equation (4), and observing that

$$l\left(\frac{z+1}{z}\right) = l(z+1) - lz, \quad \text{we find}$$

$$l(z+1) - lz = 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \&c.\right) \quad (5)$$

or, by transposition,

$$l(z+1) = lz + 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \&c.\right) \quad (6)$$

Let us make use of formula (6) to explain the method of computing a table of Napierian logarithms. It may be remarked, that it is only necessary to compute, from the formula, the logarithms of prime numbers; those of other numbers may be found by taking the sum of the logarithms of their factors.

The logarithm of 1 is 0. If now we make $z = 1$, we can find the logarithm of 2; and by means of this, if we make $z = 2$, we can find the logarithm of 3, and so on, as exhibited below.

$$l1 = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad = 0.000000;$$

$$l2 = 2\left(\frac{1}{3} + \frac{1}{3 \, . \, 3^3} + \frac{1}{5 \, . \, 3^5} + \frac{1}{7 \, . \, 3^7} \cdots\right) \quad . \; . \; . \quad = 0.693147;$$

$$l3 = 0.693147 + 2\left(\frac{1}{5} + \frac{1}{3 \, . \, 5^3} + \frac{1}{5 \, . \, 5^5} + \frac{1}{7 \, . \, 5^7} \cdots\right) = 1.098612;$$

$$l4 = 2 \times l2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad = 1.386294;$$

$$l5 = 1.386294 + 2\left(\frac{1}{9} + \frac{1}{3 \, . \, 9^3} + \frac{1}{5 \, . \, 9^5} + \frac{1}{7 \, . \, 9^7} \cdots\right) = 1.609437;$$

$$l6 = l2 + l3 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad = 1.791759;$$

$$l7 = 1.791759 + 2\left(\frac{1}{13} + \frac{1}{3 \, . \, (13)^3} + \frac{1}{5 \, . \, (13)^5} + . \, .\right) = 1.945910;$$

$l8 = l4 + l2$ . . . . $= 2.079441$;

$l9 = 2 \times l3$ . . . . $= 2.197224$;

$l10 = l5 + l2$ . . . . $= 2.302585$;

&c. &c.

In like manner, we may compute the Napierian logarithms of all numbers. Other formulas may be deduced that are more rapidly converging than the one above given, but this serves to show the facility with which logarithms may be computed.

232. **Modulus and Base.**—We can now find the values of the *modulus of the common system*, and *the base of the Napierian system.*

1°. If we make $y = 9$ in equation (1), Art. **229**, we have,

$$M \times l10 = \log 10; \quad \therefore \quad M = \frac{\log 10}{l10}.$$

But the logarithm of 10 in the common system is 1, and in the Napierian system (Art. **231**), 2.302585.

Hence,

$$M = \frac{1}{2.302585} = 0.434294482\,..$$

Which is the *Modulus of the common system.* If we multiply the Napierian logarithms already found by this value of M, the products will be the common logarithms of the same numbers.

2°. If we denote the base of the Napierian system by e, and then make $1 + y = e$, in equation (1), Art. **229**, we have,

$$M \times le = \log e.$$

But, $M = .434294482$, and from Art. **226**, supposition 2°, $le = 1$; hence,

$$\log e = 0.434294482 \ldots$$

As we have already explained the method of calculating the common tables, we may use them to find the number whose logarithm is 0.434294482, which we shall find to be 2.718281828; hence,

$$e = 2.718281828.$$

Of Interpolation.

233. When the law of a series is given, and several terms taken at equal distances are known, we may, by means of the formula,

$$T = a + nd_1 + \frac{n(n-1)}{1 \,.\, 2}d_2 + \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3}d_3 + \text{\&c.} \ldots (1)$$

already deduced (Art. **209**), introduce other terms between them, that will conform to the law of the series. This operation is called *Interpolation.*

In most cases, the law of the series is not given, but only numerical values of certain terms of the series, taken at fixed intervals; in this case we can only approximate to the law of the series, or to the value of any intermediate term, by the aid of the formula.

To illustrate the use of formula (1), in interpolating a term in a tabulated series of numbers, let us suppose that we have the logarithms of 12, 13, 14, 15, and that it is required to find the logarithm of 12½. Forming the orders of differences from the logarithms of 12, 13, 14, and 15, respectively, and taking the first terms of each,

12 13 14 15
1.079181, 1.113943, 1.146128, 1.176091,
0.034762, 0.032185, 0.029963,
− 0.002577, − 0.002222,
+ 0.000355,

we find

$$d_1 = 0.034762, \quad d_2 = -0.002577, \quad d_3 = 0.000355.$$

If we consider log 12 as the first term, we have also

$$a = 1.079181 \quad \text{and} \quad n = \frac{1}{2}.$$

Making these several substitutions in the formula, and neglecting the terms after the fourth, since they are inappreciable, we find,

$$\text{T} = a + \frac{1}{2}d_1 - \frac{1}{8}d_2 + \frac{1}{16}d_3 = \log 12\tfrac{1}{2};$$

or, substituting for d_1, d_2, &c., their values, and for a its value,

a	1.079181
$\frac{1}{2}d_1$	0.017381
$\frac{1}{8}d_2$	0.000322
$\frac{1}{16}d_3$	0.000022
Log $12\frac{1}{2}$	1.096906

Had it been required to find the logarithm of 12.39, we should have made $n = .39$, and the process would have been the same as above. In like manner we may interpolate terms between the tabulated terms of any mathematical table.

II. CONTINUED FRACTIONS.

234. The method of computing logarithms by means of series, was shown in Article **231**. The logarithms of numbers may also be computed by the method of *Continued Fractions*, which will now be explained.

A **Continued Fraction** is a fraction that has 1 for its numerator, and for its denominator a whole number plus a fraction that has 1 for its numerator and for its denominator a whole number plus a fraction, and so on.

Thus, every expression of the form

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\&c.}}}}$$

in which a, b, c, d, &c., are positive whole numbers, is a *continued fraction.*

235. The separate fractions $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, &c., that make up a continued fraction, are called *integral fractions.*

The fractions,

$$\frac{1}{a},\quad \cfrac{1}{a+\cfrac{1}{b}},\quad \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}},\ \&c.$$

are called *approximating fractions,* because each gives, in succession, a nearer approximation to the true value of the fraction.

An Approximating fraction is the result obtained by stopping at any integral fraction, and neglecting all that follows.

If we stop at the first integral fraction, the resulting approximating fraction is said to be of the *first order;* if at the second integral fraction, the approximating fraction is of the *second order,* and so on.

When the number of integral fractions is *finite,* we get the true value of the expression by considering them all; when their number is *infinite,* only an approximate value can be found.

236. The value of the *first* approximating fraction is *greater* than the true value of the continued fraction; for, the denominator a is too small by the continued fraction that follows it; hence, $\frac{1}{a}$ is too great.

The value of the *second* approximating fraction is *less* than that of the continued fraction; for, the denominator b is too small by the continued fraction that follows it; consequently, $\frac{1}{b}$, and also $a+\frac{1}{b}$, is too large; hence, $\dfrac{1}{a+\frac{1}{b}}$ is too small.

It may be shown, in like manner, that *every approximating fraction of an odd order is* GREATER, *and every approximating fraction of an even order is* LESS, *than the continued fraction.*

237. The successive approximating fractions, after the second, may be deduced from the two preceding ones by a simple law.

From Article **235**, we see that the first two approximating fractions are

$$\frac{1}{a} \quad \cdots\cdots\cdots \quad (1)$$

and

$$\frac{b}{ab+1} \quad \cdots\cdots\cdots \quad (2)$$

To deduce the *third* approximating fraction, we replace b in equation (2) by $b+\frac{1}{c}$; this gives

$$\frac{b+\frac{1}{c}}{ab+\frac{a}{c}+1}=\frac{bc+1}{(ab+1)c+a} \quad \cdots\cdots \quad (3)$$

To deduce the *fourth* approximating fraction, we replace c in equation (3) by $c+\frac{1}{d}$; making this substitution, and multiplying both terms of the resulting fractions by d, we have,

$$\frac{(bc+1)d+b}{[(ab+1)c+a]d+ab+1} \quad \ldots\ldots \quad (4)$$

and so on.

The numerator of the third approximating fraction is found by adding the numerator of the first to the product of the numerator of the second by the denominator of the third integral fraction; and the denominator is found by adding the denominator of the first to the product of the denominator of the second by the same multiplier. The fourth approximating fraction is derived from the second and third by the same law, using the denominator of the fourth integral fraction as a multiplier. The fifth is derived from the third and fourth by the same law, and so on.

238. Let it be required to find the successive approximating fractions of the continued fraction.

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{4}}}}.$$

We have, for the first and second,

$$\frac{1}{2} \quad \ldots\ldots \quad (1)$$

$$\frac{3}{7} \quad \ldots\ldots \quad (2)$$

Hence, by the rule,

$$\frac{5\times 3+1}{5\times 7+2}=\frac{16}{37} \quad \ldots\ldots \quad (3)$$

$$\frac{4\times 16+3}{4\times 37+7}=\frac{67}{155} \quad \ldots\ldots \quad (4)$$

If we subtract each approximating fraction from the preceding one, we have

$$\frac{1}{2}-\frac{3}{7}=\frac{1}{7\times 2} \quad . \quad . \quad . \quad \text{1st diff.}$$

$$\frac{3}{7}-\frac{16}{37}=-\frac{1}{7\times 37} \quad . \quad . \quad . \quad \text{2d diff.}$$

$$\frac{16}{37}-\frac{67}{155}=\frac{1}{37\times 155} \quad . \quad . \quad . \quad \text{3d diff.}$$

It may be shown that this law of differences, holds good, whatever may be the number of approximating fractions; hence,

The difference between any two consecutive approximating fractions, is equal to ± 1, *divided by the product of their denominators.*

When an approximating fraction of an even order is taken from one of an odd order, the upper sign is used: when one of an odd order is taken from one of an even order, the lower sign is used.

This ought to be the case, since we have shown that every approximating fraction of an odd order is greater than the true value of the continued fraction, whilst every one of an even order is less.

239. It was shown in Article **236**, that each approximating fraction of an odd order, exceeds the true value of the continued fraction; while each of an even order is less than it. Hence, the difference between any two consecutive approximating fractions is greater than the difference between either of them and the true value of the continued fraction. Therefore, stopping at the n^{th} approximating fraction, the result will be the true value of the fraction, to within 1 divided by the denominator of that fraction, multiplied by *the denominator of the* approximating fraction which follows

But this product is greater than the square of the denominator of the n^{th} approximating fraction; hence, *any approximating fraction differs from the true value of the continued fraction, by less than* 1 *divided by the square of its denominator.*

Thus, in the example of Article **238**, the true value of the continued fraction is $\frac{67}{155}$, and $\frac{16}{37}$ differs from it by less than $\frac{1}{(37)^2}$ or $\frac{1}{1369}$, and because this is of the third order, the difference is one of excess.

240. Exponential Equation.—Let us apply the preceding principles to the solution of the exponential equation,

$$10^x = 5 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We see that x lies between 0 and 1; let us suppose it equal to $\frac{1}{y}$; this gives

$$10^{\frac{1}{y}} = 5, \quad \text{or,} \quad 5^y = 10 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Here we see that y lies between 1 and 2; suppose it equal to $1 + \frac{1}{w}$; this substituted in (2) gives,

$$5^{1+\frac{1}{w}} = 5 \times 5^{\frac{1}{w}} = 10; \quad \therefore \quad 5^{\frac{1}{w}} = 2, \quad \text{or,} \quad 2^w = 5 \quad . \quad (3)$$

Here we see that w lies between 2 and 3; suppose it equal to $2 + \frac{1}{s}$; proceeding as before, we find,

$$4 \times 2^{\frac{1}{s}} = 5, \quad \text{or} \quad \left(\frac{5}{4}\right)^s = 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Making $s = 3 + \frac{1}{t}$, we find,

$$\left(\frac{5}{4}\right)^3 \left(\frac{5}{4}\right)^{\frac{1}{t}} = 2; \quad \therefore \quad \frac{5}{4} = \left(\frac{128}{125}\right)^t$$

Here t is between 9 and 10; assuming it to be equal to 9 we have, by substituting in the preceding expressions,

$$x = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{9}}}}.$$

The successive approximating fractions are,

$$1, \ \frac{2}{3}, \ \frac{7}{10}, \ \text{and} \ \frac{65}{93} = .698925.$$

This last value differs from the log of 5 by less than $\frac{1}{(93)^2}$

Any other exponential equation may be solved in a similar manner.

EXAMPLES.

Solve the following exponential equations:

1. $8^x = 32$. *Ans.* $x = \frac{5}{3}$.

2. $3^x = 15$. *Ans.* $x = 2.46$, to within 0.01.

3. $10^x = 3$. *Ans.* $x = 0.477$. to within 0.001.

4. $5^x = \frac{2}{3}$ *Ans.* $x = -0.25$ to within 0.01.

241. Vulgar fractions may be placed under the form or continued fractions.

Take, for example, the fraction $\frac{65}{149}$, and divide both terms by 65; the value of the fraction will not be changed, and we shall have

$$\frac{65}{149} = \frac{1}{\frac{149}{65}} = \frac{1}{2 + \frac{19}{65}}.$$

Now, if we neglect the fractional part, $\frac{19}{65}$, of the denominator, we obtain $\frac{1}{2}$ for an approximate value of the given fraction. But this value is too large, since the *denominator* used is too *small*.

If, instead of neglecting the part $\frac{19}{65}$, we replace it by 1, the approximate value is $\frac{1}{3}$, which is too small, since the denominator 3 is too *large*. Hence,

$$\frac{1}{2} > \frac{65}{149} \quad \text{and} \quad \frac{1}{3} < \frac{65}{149};$$

therefore the value of the fraction lies between $\frac{1}{2}$ and $\frac{1}{3}$.

Operating on the fraction $\frac{19}{65}$ as we did on the given fraction $\frac{65}{149}$, we obtain,

$$\frac{19}{65} = \cfrac{1}{3 + \cfrac{8}{19}};$$

hence,

$$\frac{65}{149} = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{8}{19}}}.$$

It may be shown, as before, that the given fraction lies between the limits $\frac{3}{7}$ and $\frac{4}{9}$.

III. SIMPLE AND COMPOUND INTEREST.

242. The problems relating to Simple and Compound Interest may often be simplified by employing algebraic formulas. It is often the case that the ordinary algebraic formulas may be still further simplified by the application of the methods of logarithms.

In what follows, we shall employ the following notation:

Let p denote the amount bearing interest, called the *principal;*
r " the *rate per cent.;* that is, the part of a dollar for one year's interest;
t " the time, in years, that p draws interest;
i " the interest of p dollars for t years;
S " $p +$ the interest which accrues in the time, called the *amount.*

Simple Interest.

243. Interest is money paid for the use of money. When interest is paid on the principal alone, it is called *Simple Interest.*

To find the interest of a sum p for t years, at the rate r, and also, the amount then due.

Since r denotes the part of a dollar which expresses its interest for a single year, the interest of p dollars for the same time will be expressed by pr; and for t years it will be t times as much; hence,

$$i = ptr; \quad \ldots\ldots\ldots \quad (1)$$

and for the amount due,

$$S = p + ptr = p(1 + tr) \quad \ldots\ldots \quad (2)$$

EXAMPLE.

What is the interest, and what the amount of \$365 for three years and a half, at the rate of 4 per cent. per annum? Here,

$$p = \$365;$$
$$r = \frac{4}{100} = 0.04;$$
$$t = 3.5;$$
$$i = ptr = 365 \times 3.5 \times 0.04 = 51.10;$$

hence, $S = 365 + 51.10 = \$416.10.$

Present Value and Discount at Simple Interest.

244. The **Present Value** of a sum S, due t years hence, is the principal p, which put at interest for the time t, will produce the amount S.

The **Discount** on any sum due t years hence, is the difference between that sum and the present value.

To find the present value of a sum of dollars denoted by S, *due t years hence, at simple interest, at the rate r;* also, *the discount.*

We have, from formula (2),

$$S = p + ptr;$$

and since p is the principal which in t years will produce the sum S, we have,

$$p = \frac{S}{1+tr}; \quad \text{. . . . .} \quad (3)$$

for the discount, denoted by D, we have,

$$D = S - \frac{S}{1+tr} = \frac{Str}{1+tr} \quad \text{. . . . . .} \quad (4)$$

EXAMPLE.

Required the discount on \$100, due 3 months hence, at the rate of $5\frac{1}{2}$ per cent. per annum.

$$\begin{aligned} S &= \$100 \\ t &= 3 \text{ months} = 0.25, \\ r &= \frac{5.5}{100} \qquad = 0.55. \end{aligned}$$

Hence, the present value p is

$$p = \frac{S}{1+tr} = \frac{100}{1+.01375} = \$98.643;$$

and

$$D = S - p = 100 - 98.643 = \$1.357.$$

Compound Interest.

245. **Compound Interest** is when the interest on a sum of money becoming due, and not paid, is added to the principal, and the interest then calculated on this amount as on a new principal.

To find the amount of a sum p placed at interest for t years, compound interest being allowed annually at the rate r.

At the end of one year, the amount will be,

$$S = p + pr = p(1 + r).$$

This sum now becomes the principal, and hence, at the end of the second year, the amount will be,

$$S' = p(1 + r) + pr(1 + r) = p(1 + r)^2.$$

Regarding $p(1 + r)^2$ as a new principal, we have, at the end of the third year,

$$S'' = p(1 + r)^2 + pr(1 + r)^2 = p(1 + r)^3;$$

and at the end of t years, we have,

$$S = p(1 + r)^t \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

From Articles **221** and **222** we have,

$$\log S = \log p + t \log(1 + r).$$

If any three of the four quantities S, p, t, and r, are given, the remaining one can be determined.

EXAMPLE.

Let it be required to find the time in which a sum p will double itself at compound interest, the rate being 4 per cent. per annum.

We have, from equation (5),

$$S = p(1+r)^t.$$

But by the conditions of the question,

$$S = 2p = p(1+r)^t;$$

hence,

$$2 = (1+r)^t.$$

By logarithms, we have,

$$t = \frac{\log 2}{\log(1+r)} = \frac{\log 2}{\log(1.04)} = \frac{1.301030}{0.017033}$$

$$\therefore \quad t = 17.673.$$

That is, the term required is 17 *yrs.*, 8 *mos.*, 2 *days.*

Present Value and Discount at Compound Interest.

246. The **Present Value**, is a sum which, placed at compound interest for a given time, will produce a given amount. The discount is the difference between the amount and the present value.

From formula (5), we have, for the present value,

$$p = \frac{S}{(1+r)^t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For the discount, denoted by D, we have,

$$D = S - \frac{S}{(1+r)^t} = S\left(1 - \frac{1}{(1+r)^t}\right) \quad . \quad . \quad . \quad (7)$$

In this and in the preceding articles, we have supposed the unit of time to be one year, and the rate, per year. If interest is made payable at smaller intervals of time, say n times per year, the formulas from (1) to (7) will hold true, provided we make t equal to the number of *periods*, and r equal to the annual rate divided by the number of *periods* in a year. Thus, if interest is made payable quarterly,

t will denote the number of *quarter years*, and r will be *one fourth* the annual rate.

IV. Permutations, Arrangements, and Combinations.

Permutations.

247. Permutations, are the results obtained by writing a certain number of letters one after another, in every possible order, in such a manner that all the letters shall enter each result, and each letter but once.

To find the number of permutations of n letters, n being any positive whole number, let us begin with two letters.

Two letters, a and b, evidently give two permutations.

$$\left\{\begin{array}{l} ab \\ ba \end{array}\right.$$

hence, the number of permutations is 1×2.

Next, let us take three letters, a, b, and c. Reserve either of the letters, as c, and permute the other two, giving

$$\left\{\begin{array}{ll} c & \\ & ab \\ & ba \end{array}\right.$$

Now, the third letter c may be placed before ab, between a and b, and at the right of ab; and the same for ba: that is, in *one of the first permutations, the reserved letter* c *may have three different places, giving three permutations.* And, as the same may be shown for *each* of the first permutations, it follows that the whole number of permutations, of three letters, is $1 \times 2 \times 3$.

$$\left\{\begin{array}{l} cab \\ acb \\ abc \\ cba \\ bca \\ bac \end{array}\right.$$

If a fourth letter d be introduced, it can have four places in *each one* of the six permutations of three letters; hence, the number of permutations of four letters is, $1 \times 2 \times 3 \times 4$.

If a fifth letter, h, be introduced, it can have five places in each of the twenty-four permutations of four letters; hence, the number of permutations of five letters is $1 \times 2 \times 3 \times 4 \times 5$; and so on, indefinitely.

Hence, if Y denote the number of permutations of n letters, we shall have

$$Y = 1.\ 2.\ 3.\ 4.\ .\ .\ .\ (n-1)n: \text{ that is,}$$

The number of permutations of n letters, is equal to the continued product of the natural numbers from 1 to n inclusive.

In the preceding formula, and in other cases where there is no danger of confounding it with the decimal point, the sign of multiplication ordinarily used between numerical factors is replaced by a simple period or dot.

Arrangements.

248. Arrangements, are the results obtained by writing m letters, in sets of 2 and 2, 3 and 3, 4 and 4, ... n and n; the letters in each set having every possible order, and m being greater than n.

To find the number of arrangements of m letters taken, n in a set, let us first arrange three letters, a, b, and c, in sets of two.

First, arrange them in sets of one, and for each set so formed, there will be two letters reserved, the reserved letters for each arrangement, being those which do not enter it. Thus, with reference to a, the reserved letters are b and c; with reference to b, the reserved letters are a and c; and with reference to c, they are a and b.

$$\left\{ \begin{array}{c} a \\ b \\ c \end{array} \right.$$

Now, to any one of the letters, as a, annex, in succession, the reserved letters b and c; to the second arrangement, b, annex the reserved letters a and c; and to the third arrangement, c, annex the reserved letters a and b.

$$\left\{\begin{array}{cc} a & b \\ a & c \\ b & a \\ b & c \\ c & a \\ c & b \end{array}\right.$$

Since each of the first arrangements gives as many new arrangements as there are reserved letters, it follows,

That the number of arrangements of three letters taken, two in a set, is equal to the number of arrangements of the same letters taken one in a set, multiplied by the number of reserved letters.

If we take m letters and arrange them one in a set, we shall have m arrangements, and $m-1$ reserved letters. If we annex each of the reserved letters to each of the m arrangements, we shall have $m(m-1)$ new arrangements in sets of two, and for each arrangement there will be $m-2$ reserved letters. In like manner, if we annex each of the reserved letters to each of the $m(m-1)$ arrangements, we shall have $m(m-1)(m-2)$ new arrangements, in sets of three; and so on. Hence, if we denote the number of m letters taken n in a set by X, we shall have,

$$X = m(m-1)(m-2)\ .\ .\ .\ (m-n+1); \text{ that is,}$$

The number of arrangements of m letters, taken n in a set, is equal to the continued product of the natural numbers from m down to $m-n+1$, inclusive.

If in the preceding formula, m be made equal to n, the arrangements become permutations, and the formula reduces to

$$X = n(n-1)(n-2)\ .\ .\ .\ .\ 2\ .\ 1;$$

or, by reversing the order of the factors, and writing Y for X,

$$Y = 1\ .\ 2\ .\ 3\ .\ .\ .\ .\ (n-1)n;$$

the same formula as deduced in the last article.

Combinations.

249. Combinations, are arrangements in which any two differ from each other by at least one of the letters that enter them.

To deduce a formula for the number of combinations of m letters, taken n in a set, let X denote the number of arrangements of m letters, taken in sets of n, Y the number of permutations of n letters, and Z the number of *combinations* taken in sets of n.

It is evident, that all the possible arrangements of m letters, taken n in a set, can be obtained by subjecting the n letters of each of the Z combinations, to all the *permutations* of which these letters are susceptible. Now, a single combination of n letters gives, by hypothesis, Y permutations; therefore, Z combinations gives Y × Z arrangements; and as X denotes the total number of arrangements, it follows that

$$X = Y \times Z; \qquad \text{whence,} \qquad Z = \frac{X}{Y}.$$

But we have (Art. **247**),

$$Y = Q \times n = 1 . 2 . 3 \ldots . n,$$

and (Art. **248**),

$$X = m(m-1)(m-2) \ldots . (m-n+1);$$

therefore,

$$Z = \frac{m(m-1)(m-2) \ldots . (m-n+1)}{1 . 2 . 3 \ldots \ldots . n}$$

that is,

The number of combinations of m letters taken n in a set, is equal to the continued product of the natural numbers from m down to $m - n + 1$ *inclusive, divided by the continued product of the natural numbers from* 1 *to n inclusive.*

250. If Z denote the number of combinations of the m letters taken n in a set, we have just seen that

$$Z = \frac{m(m-1)(m-2)\dots(m-n+1)}{1.2.3\dots\dots n} \quad . \quad . \quad (1)$$

If Z′ denote the number of combinations of m letters taken $(m-n)$ in a set, we can find an expression for Z′ by changing n into $m-n$ in the above formula; whence,

$$Z' = \frac{m(m-1)(m-2)\dots(n+1)}{1.2.3\dots\dots(m-n)} \quad . \quad . \quad . \quad (2)$$

If we divide equation (1) by (2), member by member, and arrange the factors of both terms of the quotient, we have,

$$\frac{Z}{Z'} = \frac{1.2.3\dots(m-n)\times(m-n+1)\dots(m-1)m}{1.2.3\dots\dots n\times(n+1)\dots\dots\dots(m-1)m}.$$

The numerator and denominator of the second member are equal; hence,

$$\frac{Z}{Z'} = 1, \quad \text{or} \quad Z = Z'; \quad \text{therefore,}$$

The number of combinations of m letters, taken n in a set, is equal to the number of combinations of m letters, taken m − n in a set.

EXAMPLES.

1. In how many ways can 6 boys be seated in class so as not to be placed in the same order?

Here, $n = 1$, and from Art. **247** we have,

$$Y = 1.2\dots6 = 720. \quad \textit{Ans.}$$

2. How many changes can be rung on 10 bells, taken in sets of 7?

Here, $m = 10$, $n = 7$, and from Art. **248** we have,

$$X = 10 \,.\, 9 \ldots . 4 = 604800.$$

3. How many combinations, 3 in a set, can be formed from 5 letters? *Ans.* $\dfrac{5 \,.\, 4 \,.\, 3}{1 \,.\, 2 \,.\, 3} = 10.$

V. PROBABILITIES.

251. When there are m chances that an event will *happen*, and n chances that it will *fail*, and when there is no reason that one of these chances should occur rather than another, the *probability* that the event will happen is $\dfrac{m}{m+n}$, and the *probability* that it will fail is $\dfrac{n}{m+n}$.

The sum of these probabilities is 1; and when an event must either *happen* or *fail*, we may regard 1 as the measure of certainty. Hence, if p denotes the probability of the occurrence of an event, $1-p$ will denote the probability of its failure.

Simple Probabilities.

252. When two events are so related that one must happen, and but one can happen, if the first event can happen in m ways, and the second in n ways, and if all these ways are equally likely to occur, the probability that the first event will happen is $\dfrac{m}{m+n}$, and the probability that the second will happen is $\dfrac{n}{m+n}$. Thus, if three black balls and two white ones are placed in an urn, and one drawn out at random, the chance that it will be a black one is $\dfrac{3}{5}$, and the chance that it will be a white one

Here, the sum of the probabilities is 1, or certainty.

When there are three events, so connected that one must happen, and but one, if the first can happen in a ways, the second in b ways, and the third in c ways, the probability that the first will happen is $\frac{a}{a+b+c}$, the probability that the second will happen is $\frac{b}{a+b+c}$, and the probability that the third will happen is $\frac{c}{a+b+c}$. Thus, if 3 white balls, 4 black balls, and 5 red ones, be placed in an urn, and one drawn at random, the probability that it will be white is $\frac{3}{12}$, the probability that it will be black is $\frac{4}{12}$, and the probability that it will be red is $\frac{5}{12}$. Here, again, the sum of the probabilities is 1, or certainty.

From what precedes we may write the following rule for determining the probability of the occurrence of a contingent event.

RULE.

Divide the number of favorable chances by the whole number of chances, both favorable and unfavorable.

Complex Probabilities.

253. To determine the probability of the simultaneous occurrence of two independent events, let us suppose there are m ways in which the first may occur, and n ways in which it may fail, and that there are m' ways in which the second may occur, and n' ways in which it may fail. Each of the $m+n$ ways, in the first case, may be combined with each of the $m'+n'$ ways, in the second case, giving $(m+n)(m'+n')$ chances, in all. If we combine each of the m ways in the first case with each of the m' ways in the second case, we have mm' chances for the simultaneous occurrence of the two events; hence, according to the rule, the probability of the simultaneous occurrence of the two events in question is,

$$\frac{mm'}{(m+n)(m'+n')}=\frac{m}{m+n}\times\frac{m'}{m'+n'}.$$

That is, *it is equal to the product of the probabilities that each will occur separately.*

This principle may be extended by the same course of reasoning to the determination of the probability of the simultaneous occurrence of any number of independent events, giving the following

RULE.

The probability of the simultaneous occurrence of any number of independent events, is equal to the continued product of the probabilities that each will occur separately.

EXAMPLES.

1. What is the probability of throwing three aces, with three dice, at a single throw?

The probability that each die will turn up an ace, is $\frac{1}{6}$. Hence, the probability that all will turn up aces together is $\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{1}{216}$, *Ans.*

2. What is the probability that neither of the three will turn up an ace?

The probability that any one will not turn up an ace is $\frac{5}{6}$. Hence, the probability that neither of the three will turn an ace, is $\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{125}{216}$.

3. What are the respective probabilities that two of the dice will turn up aces and one not, and that one will turn up ace and the other two not?

The probability that the first two will be aces is $\frac{1}{6} \times \frac{1}{6}$, or $\frac{1}{36}$, and the probability that the third will not be an ace is $\frac{5}{6}$. Hence, the probability that the first and second will be aces and the third not, is $\frac{1}{36} \times \frac{5}{6}$, or $\frac{5}{216}$. In like manner the probability that the first and third will be aces and the second not, is $\frac{5}{216}$, and the probability that the second and third will be aces and the first not, is $\frac{5}{216}$. Hence, the total probability that two will be aces and the other one not, is $\frac{15}{216}$.

In like manner, the probability that one will be an ace and the other two not, is $3 \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{75}{216}$.

The sum of the probabilities in all these cases is,

$$\frac{1}{216} + \frac{125}{216} + \frac{15}{216} + \frac{75}{216} = 1, \text{ as it should be.}$$

The preceding rule covers the case of successive independent events. If a single die be thrown three times, in succession, and the results recorded, it is obvious that the same laws of probability would obtain, as though the three dice had been thrown at once.

254. The preceding principles enable us to solve most of the ordinary problems of probability. They will be illustrated by the solution of one or two examples.

1. Suppose 3 *white* balls, 4 *black* balls, and 5 *red* balls to be thrown into an urn, and that two are drawn out at random; what are the probabilities of the different cases that may arise?

By the law of combination (Art. **249**), we find that the twelve balls may be combined in sets of 2, that is, in pairs, in $\frac{11 \times 12}{1 \times 2}$, or in 66 different ways; the 3 white balls may be combined so as to give 3 different pairs; the 4 black balls so as to give 6 pairs; the 5 black balls so as to give 10 pairs. Further, each white ball may be combined with each black one, giving 12 pairs; and in like manner we have 15 pairs, one white and one red; and 20 pairs, one red and one black. Hence, the probability of drawing

Two white balls, is	$\frac{3}{66}$;
Two black balls, is	$\frac{6}{66}$;
Two red balls, is	$\frac{10}{66}$;
One white and one black, is	$\frac{12}{66}$;
One white and one red, is	$\frac{15}{66}$;
One black and one red, is	$\frac{20}{66}$.

The sum of these probabilities is equal to 1, as it ought to be.

2. According to "American Experience," table of mortality, of 847 persons of the age of 90, only 462 reach the age of 91, and only 216 reach the age of 92. What are the robabilities that one person aged 90 and one aged 91 will, h be living or both be dead at the end of one year?

The probability that the *first* will live one year is $\frac{462}{847}$, and the probability that he will not be living is $\frac{385}{847}$; the probability that the second will live one year is $\frac{216}{462}$ or $\frac{108}{231}$, and the probability that he will not live one year is $\frac{123}{231}$.

Hence, the probability that both will be alive at the end of the year is $\frac{462 \times 108}{847 \times 231}$, or a little more than $\frac{1}{4}$; and the probability that neither will be alive at the end of the year is $\frac{385}{847} \times \frac{123}{231}$, or a little less than $\frac{1}{4}$. Hence, the probability that both will be alive at the end of the year is a little greater than that neither will be living.

The principles of probability are used in solving the various problems of annuities and life insurance; they are also used in determining the most reliable results springing from discordant observations in physics and astronomy.

CHAPTER X.

GENERAL THEORY OF EQUATIONS.

I. PROPERTIES OF EQUATIONS.

255. Every equation containing but one unknown quantity, whose exponents are whole numbers, can be reduced to the form

$$x^m + Px^{m-1} + Qx^{m-2} + \dots + Tx + U = 0 \quad . \quad . \, (1)$$

In this equation P, Q, ... T, U, are coefficients in the most general sense of the term; that is, they may be positive or negative, entire or fractional, real or imaginary. Equation (1) is called *the reduced equation.* The method of making the reduction is analogous to that given for reducing equations of the second degree to the form (Art. **120**),

$$x^2 + 2px - q = 0.$$

In equation (1) the term U, which is the coefficient of x^0, is called the *absolute* term.

If, in the reduced equation (1), none of the coefficients are 0, the equation is said to be *complete;* if one or more of the coefficients are zero, the equation is *incomplete.*

In discussing the properties of equations of the m^{th} degree, containing but one unknown quantity, we shall suppose them to have been reduced to the above form.

256. In defining an equation to be an expression of ᐧlity between two quantities, it is assumed that the expression may be verified by *some* value of the unknown quantity, either real or imaginary; such a value is called a *root* of the equation. Hence, we infer that *every equation of the form* (1) *has at least one root.*

Properties of Equations.

1°. *If a is a root of the reduced equation, then is the first member of that equation exactly divisible by* $x - a$.

Let us apply the rule for the division of the first member by $x - a$, and continue the operation till a remainder is found that is independent of x; that is, which does not contain x.

Denote this remainder by R, and represent the quotient found by Q′, and we shall have,

$$x^m + Px^{m-1} \dots + Tx + U = Q'(x - a) + R \quad . \quad . \quad (2)$$

Now, by hypothesis, a is a root of the equation; if, therefore, we substitute a for x, the first member will reduce to zero; the term $Q'(x - a)$ will also reduce to 0, and consequently, we shall have,

$$R = 0.$$

But R does not contain x; hence, it is equal to 0, whatever may be the value of x, and consequently, the first member of the equation

$$x^m + Px^{m-1} + Qx^{m-2} \dots + Tx + U = 0,$$

is exactly divisible by $x - a$.

2°. Conversely, *if the first member of the reduced equation is exactly divisible by* $x - a$, *then is* a, *a root of the equation.*

For, in this case, the remainder R in equation (2) is equal to 0; hence, that equation becomes

$$x^m + Px^{m-1} + \dots + Tx + U = Q'(x - a) \quad . \quad . \quad (3)$$

If in (3) we make $x = a$, the second member reduces to 0, and consequently the first member also reduces to 0; that is, making $x = a$ satisfies equation (1); hence, a is a root of that equation.

It is evident, from the nature of division, that the quotient Q′ will be of the form

$$x^{m-1} + P'x^{m-2} \ldots\ldots\ldots + R'x + U' = 0.$$

It follows from what precedes, that in order to discover whether any polynomial is exactly divisible by the binomial $x - a$, it is sufficient to see whether the substitution of a for x, reduces the polynomial to zero.

Conversely, when a polynomial is exactly divisible by $x - a$, we know, that if the polynomial be placed equal to zero, a will be a root of the resulting equation.

Property 1° enables us to diminish the degree of an equation by 1 when we know one of its roots, by simple division; if two or more roots are known, the degree of the equation may be still further diminished by successive divisions.

EXAMPLES.

1. A root of the equation,

$$x^4 - 25x^2 + 60x - 36 = 0,$$

is 3: what does the equation become when freed of this root?

$$\begin{array}{l|l}
x^4 - 25x^2 + 60x - 36 & x - 3 \\ \cline{2-2}
\underline{x^4 - 3x^3} & x^3 + 3x^2 - 16x + 12. \\
+ 3x^3 - 25x^2 & \\
\underline{3x^3 - 9x^2} & \\
- 16x^2 + 60x & \\
\underline{- 16x^2 + 48x} & \\
12x - 36 & \\
\underline{12x - 36} & \\
\end{array}$$

Ans. $x^3 + 3x^2 - 16x + 12 = 0.$

2. Two roots of the equation,

$$x^4 - 12x^3 + 48x^2 - 68x + 15 = 0,$$

are 3 and 5: what does the equation become when freed of them? *Ans.* $x^2 - 4x + 1 = 0$.

3. A root of the equation,

$$x^3 - 6x^2 + 11x - 6 = 0,$$

is 1: what is the reduced equation? *Ans.* $x^2 - 5x + 6 = 0$.

4. Two roots of the equation,

$$4x^4 - 14x^3 - 5x^2 + 31x + 6 = 0,$$

are 2 and 3: find the reduced equation. *Ans.* $4x^2 + 6x + 1 = 0$.

3°. *Every equation containing but one unknown quantity, has as many roots as there are units in the exponent that denotes its degree, and no more.*

Let the proposed equation be

$$x^m + Px^{m-1} + Qx^{m-2} + \dots + Tx + U = 0.$$

Since every equation has at least one root, if we denote that root by a, the first member will be divisible by $x - a$, and we shall have the equation,

$$x^m + Px^{m-1} + \dots = (x - a)(x^{m-1} + P'x^{m-2} + \dots) \quad . \quad (4)$$

But if we place,

$$x^{m-1} + P'x^{m-2} + \dots = 0,$$

we obtain a new equation, which has at least one root.

Denote this root by b, and we have

$$x^{m-1} + P'x^{m-2} + \dots = (x - b)(x^{m-2} + P''x^{m-3} + \dots).$$

Substituting the second member, for its value, in equation (4), we have,

$$x^m + Px^{m-1} + \ldots = (x-a)(x-b)(x^{m-2} + P''x^{m-2} + \ldots) . \quad (5)$$

Reasoning on the polynomial,

$$x^{m-2} + P''x^{m-3} + \ldots,$$

as on the preceding polynomial, we have

$$x^{m-2} + P''x^{m-3} + \ldots = (x-c)(x^{m-3} + P'''x^{m-4} + \ldots),$$

and by substitution,

$$x^m + Px^{m-1} + \ldots = (x-a)(x-b)(x-c)(x^{m-3} + P'''x^{m-4}) \quad (6)$$

By continuing this operation, we see that for each binomial factor of the first degree with reference to x, that we remove, the degree of the polynomial factor is reduced by 1; therefore, after $m-2$ binomial factors have been removed, the polynomial factor will become of the second degree with reference to x, which can be decomposed into two factors of the first degree (Art. **124**), of the form $x-k$, $x-l$.

Now, supposing the $m-2$ factors of the first degree to have already been indicated, we shall have the identical equation,

$$x^m + Px^{m-1} + \ldots = (x-a)(x-b)(x-c) \ldots (x-k)(x-l) = 0; \quad (7)$$

from which we see, that the *first member of the proposed equation has been resolved into m binomial factors of the first degree.*

As there is *a root* corresponding to each binomial factor, it follows that the m binomial factors of the first degree, $x-a$, $x-b$, $x-c$, ..., give the m roots, a, b, c, . . ., of the proposed equation.

But the equation can have no other roots than a, b, c

... k, l. For, if it had a root a', different from $a, b, c \ldots l$, it would have a divisor $x - a'$, different from $x - a$, $x - b$, $x - c \ldots x - l$, which is impossible; therefore,

Every equation of the m^{th} degree has m roots, and can have no more.

257. In equations that arise from the multiplication of equal factors, such as

$$(x - a)^4 (x - b)^3 (x - c)^2 (x - d) = 0,$$

the number of roots is *apparently* less than the exponent that denotes the degree of the equation. But this is not really so; for the above equation actually has ten roots, four of which are equal to a, three to b, two to c, and one to d.

It is evident that no quantity a', different from a, b, c, d, can verify the equation; for, if it had a root a', the first member would be divisible by $x - a'$, which is impossible.

258. It has been shown that the first member of every equation of the m^{th} degree, has m binomial divisors of the first degree, of the form

$$x - a, \quad x - b, \quad x - c, \ldots x - k, \quad x - l.$$

If we multiply these divisors together, *two and two, three and three*, &c., we shall obtain as many divisors of the second, third, &c., degree, with reference to x, as we can form combinations of m quantities, taken two and two, three and three, &c. Now, the number of these combinations is

$$m \cdot \frac{m-1}{2}, \; m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3}, \quad \text{\&c. (Art. 249.)}$$

hence, the given equation has

$$m \cdot \frac{m-1}{2}$$

divisors of the second degree;

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3}$$

divisors of the third degree;

$$m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4}$$

divisors of the fourth degree; and so on.

Composition of Equations.

259. When we know the roots of an equation we can construct the equation by subtracting each root from the unknown quantity, multiplying the resulting binomials together, and then placing the continued product equal to 0. This product may be found by actual multiplication, or it may be written from two general laws that will now be deduced. If we form the products of $x-a$, $x-b$, $x-c$, and $x-d$ by actual multiplication, we shall have,

$$\begin{array}{l} x-a \\ x-b \end{array}$$

1st product . . . $$\begin{array}{r|l} x^2-a & x+ab \\ -b & \end{array}$$

$$x-c$$

2d product . . . $$\begin{array}{r|r|l} x^3-a & x^2+ab & x-abc \\ -b & +ac & \\ -c & +bc & \end{array}$$

$$x-d$$

3d product . . . $$\begin{array}{r|r|r|l} x^4-a & x^3+ab & x^2-abc & x+abcd. \\ -b & +ac & -abd & \\ -c & +bc & -acd & \\ -d & +ad & -bcd & \\ & +bd & & \\ & +cd & & \end{array}$$

Examining these products, we see that they conform to two general laws.

1°. **The Law of Exponents.**—*The exponent of the unknown quantity in the first term is equal to the number of factors; and the exponent of that quantity goes on diminishing by* 1 *in each term toward the right, to the last term, where it is* 0.

2°. **Law of Coefficients.**—*The coefficient of the first term is* 1; *the coefficient of any term after the first is equal to the sum of all the different products of the second terms of the binomials, taken as many in a set, as there are preceding terms.*

It may be shown, by a course of reasoning entirely analogous to that employed in Article **141**, that these laws hold true for the formation of the continued product of any number of binomial factors of the proposed form.

If we resume the identical equation (7) of Article **256**, which is,

$$x^m + Px^{m-1} + Qx^{m-2} \dots + U = (x-a)(x-b)(x-c) \dots (x-l) \dots$$

and suppose the multiplications indicated, in the second member, to be performed, we shall have, from the laws just deduced, the following relations:

$$\begin{aligned} P &= -a-b-c- \dots -k-l, \\ Q &= ab+ac+bc+ \dots ak+kl, \\ R &= -abc-abd-bcd \dots -ikl, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ U &= \pm abcd \dots ikl. \end{aligned}$$

Hence, we may form the coefficients by the following

RULE.

I. Change the signs of all the roots, and take the algebraic sum of the results for the coefficient of the second term:

II. Form all the different products of the results, taken two and two, and take their algebraic sum for the coefficient of the third term:

III. Form all the products of the results, taken three in a set, and take their algebraic sum for the coefficient of the fourth term; and so on:

IV. Form the continued product of all the results for the absolute term.

The absolute term is divisible by each of the roots. If one root is 0, the absolute term is 0; and conversely, if the absolute term is 0, one of the roots is 0.

EXAMPLES.

1. Find the equation whose roots are 2, 3, 5, and -6.

We have, from the principles already established, the equation

$$(x-2)(x-3)(x-5)(x+6)=0;$$

whence, by the application of the preceding principles, we obtain the equation,

$$x^4-4x^3-29x^2+156x-180=0.$$

2. Find the equation whose roots are 1, 2, and -3.
Ans. $x^3-7x+6=0$.

3. Find the equation whose roots are 3, -4, $2+\sqrt{3}$, and $2-\sqrt{3}$. *Ans.* $x^4-3x^3-15x^2+49x-12=0$.

4. Find the equation whose roots are $3+\sqrt{5}$, $3-\sqrt{5}$, d -6. *Ans.* $x^3-32x+24=0$.

5. Find the equation whose roots are 1, -2, 3, -4, 5, and -6.

Ans. $x^6 + 3x^5 - 41x^4 - 87x^3 + 400x^2 + 444x - 720 = 0$.

6. Find the equation whose roots are . . . . $2 + \sqrt{-1}$, $2 - \sqrt{-1}$, and -3. *Ans.* $x^3 - x^2 - 7x + 15 = 0$.

II. TRANSFORMATION OF EQUATIONS.

260. The **Transformation** of an equation is the operation of changing its form without destroying the equality of its members.

261. *To transform a reduced equation containing fractional coefficients into one of the same form, whose coefficients are entire.*

Let us take the equation,

$$x^4 + \frac{a}{b}x^3 + \frac{c}{d}x^2 + \frac{e}{f}x + \frac{g}{h} = 0 \quad . \quad . \quad . \quad . \quad (1)$$

and in it make $x = \frac{y}{k}$; there will result the equation,

$$\frac{y^4}{k^4} + \frac{ay^3}{bk^3} + \frac{cy^2}{dk^2} + \frac{ey}{fk} + \frac{g}{h} = 0 \quad . \quad . \quad . \quad . \quad (2)$$

Multiplying both members of (2) by k^4, we have,

$$y^4 + \frac{ak}{b}y^3 + \frac{ck^2}{d}y^2 + \frac{ek^3}{f}y + \frac{gk^4}{h} = 0 \quad . \quad . \quad . \quad (3)$$

Since k is arbitrary, we may give to it such a value as will make the coefficients of the different powers of y entire. This can always be done; for, if we make k equal to the least common multiple of all the denominators of equation (1), the different powers of k will be divisible by each denominator separately, and, consequently, the coefficients will be entire.

EXAMPLES.

1. Transform the equation,

$$x^3 - \frac{7}{3}x^2 + \frac{11}{36}x - \frac{25}{72} = 0 \quad . \; . \; . \; . \quad (1)$$

into one whose coefficients are entire.

Making $x = \frac{y}{k}$, and multiplying both members by k^3, we have

$$y^3 - \frac{7k}{3}y^2 + \frac{11k^2}{36}y - \frac{25k^3}{72} = 0 \quad . \; . \; . \; . \quad (2)$$

Making $k = 72$, and reducing, we have,

$$y^3 - 168y^2 + 1584y - 129600 = 0 \; . \; . \; . \; . \quad (3)$$

In this and similar cases, we may often find simpler results by giving to k such a value as will cause k, k^2, k^3, &c., to contain all the prime factors that enter the denominators, each raised to a power at least equal to that of the same factor in the corresponding denominator. Thus, equation (2) may be written

$$y^3 - \frac{7k}{3}y^2 + \frac{11k^2}{2^2 \cdot 3^2}y - \frac{25k^3}{2^3 \cdot 3^2} = 0 \quad . \; . \; . \quad (4)$$

Making $k = 2 \times 3$, and reducing, we have,

$$y^3 - 14y^2 + 11y - 75 = 0.$$

Transform the following equations so as to free them from fractions, without changing the coefficient of the first term:

2. $$x^4 - \frac{5}{6}x^3 + \frac{5}{12}x^2 - \frac{7}{150}x - \frac{13}{9000} = 0.$$

Making $x = \frac{y}{k}$, and multiplying by k^4, and factoring, we have,

$$y^4 - \frac{5k}{2 \cdot 3}y^3 + \frac{5k^2}{2^2 \cdot 3}y^2 - \frac{7k^3}{2 \cdot 3 \cdot 5^2}y - \frac{13k^4}{2^3 \cdot 3^2 \cdot 5^3} = 0.$$

Making $k = 2 . 3 . 5$, and reducing, we have,

$$y^4 - 25y^3 + 375y^2 - 1260y - 1170 = 0.$$

3. $$x^3 - \frac{41}{30}x^2 + \frac{1}{3}x - \frac{1}{30} = 0.$$

Making $k = 3 . 2 . 5$, and reducing, we have,

$$y^3 - 41y^2 + 300y - 900 = 0.$$

4. $$x^3 + \frac{1}{4}x^2 + \frac{1}{16}x + \frac{1}{64} = 0.$$

Ans. $y^3 + y^2 + y + 1 = 0.$

5. $$x^5 - \frac{13}{12}x^4 + \frac{21}{40}x^3 - \frac{32}{225}x^2 - \frac{43}{600}x - \frac{1}{800} = 0.$$

Making

$$x = \frac{y}{2^2 . 3 . 5} = \frac{y}{60},$$

and we have,

$$y^5 - 65y^4 + 1890y^3 - 30720y^2 - 928800y + 972000 = 0.$$

Having solved the transformed equation, and found the values of y, the corresponding values of x may be found from the equation,

$$x = \frac{y}{k},$$

by substituting for y and k their proper values.

262. *To transform the reduced equation into another of the same form, whose roots are equal to any multiple of those of the given equation.*

If we make $x = \frac{y}{k}$ in the reduced equation, and then multiply both members of the resulting equation by k^n, we have,

$$y^n + Pky^{n-1} + Qk^2y^{n-2} + \ldots + Tk^{n-1}y + Uk^n = 0 \quad . \quad . \quad (1).$$

which is the required equation, because in it $y = kx$; k may be positive or negative, entire or fractional.

If, $k = -1$, we have, from equation (1),

$$y^n - Py^{n-1} + Qy^{n-2} + \ldots \pm Ty \mp U = 0 \quad . \quad . \quad (2)$$

in which the roots are equal to those of equation (1) with their signs changed. The upper signs of the last two terms are to be used when n is odd, and the lower ones when n is even.

EXAMPLES.

1. Transform the equation,

$$x^3 - 3x^2 + 11x + 4 = 0,$$

into one whose roots are twice as great.

Making $x = \frac{y}{2}$, and multiplying by 8, we have,

$$y^3 - 6y^2 + 44y + 32 = 0.$$

2. Transform the equation,

$$x^2 + 11x + 28 = 0$$

into one whose roots are 3 times as great.

Ans. $y^2 + 33y + 152 = 0.$

3. Transform the equation

$$x^5 + 3x^4 - 4x^3 + x^2 - x + 4 = 0$$

into one whose roots are equal to those of the given equation with their signs changed.

Ans. $y^5 - 3y^4 - 4y^3 - y^2 - y - 4 = 0.$

263. *To transform the reduced equation into one of the same form, whose roots are the reciprocals of those of the given equation.*

If we make $x = \frac{1}{y}$ in the reduced equation, we shall have,

$$\frac{1}{y^m} + \frac{P}{y^{m-1}} + \frac{Q}{y^{m-2}} + \ldots + \frac{S}{y^2} + \frac{T}{y} + U = 0 \quad . \quad (1)$$

If we multiply both members of equation (1) by $\frac{y^m}{U}$ and then reverse the order of the terms, we have,

$$y^m + \frac{T}{U}y^{m-1} + \frac{S}{U}y^{m-2} + \ldots + \frac{Q}{U}y^2 + \frac{P}{U}y + \frac{1}{U} = 0,$$

which is the required equation. We see that it is found by reversing the order of the coefficients, and dividing each by the absolute term.

EXAMPLES.

Transform the following equations into others whose roots are the reciprocals of those of the given equations:

1. $x^3 + 3x^2 + x + \frac{1}{3} = 0.$

Ans. $y^3 + 3y^2 + 9y + 3 = 0.$

2. $x^4 + x^3 + 3x + 2 = 0.$

Ans. $y^4 + \frac{3}{2}y^3 + \frac{1}{2}y + \frac{1}{2} = 0.$

3. $x^2 + \frac{7}{2}x - \frac{7}{2} = 0.$

Ans. $y^2 - y - \frac{2}{7} = 0.$

264. *To transform the reduced equation into another of the same form, whose roots shall differ from those of the given equation by any given quantity.*

Writing out the reduced equation, we have,

$$x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U = 0 \quad . \quad . \quad (1)$$

Making $x = y + x'$, in which y is a new unknown quantity and x' entirely arbitrary, this equation becomes

$$(y+x')^m + P(y+x')^{m-1} + Q(y+x')^{m-2} \ldots + T(y+x') + U = 0 \quad (2)$$

Developing by the binomial formula and arranging with respect to y, we have,

$$\left.\begin{array}{l|c|r} y^m + mx' & y^{m-1} + m\,.\,\frac{m-1}{2}x'^2 & y^{m-2} + \ldots + x'^m \\ \quad + P & + (m-1)Px' & + Px'^{m-1} \\ & + Q & + Qx'^{m-2} \\ & & + \ldots \\ & & \ldots\ldots \\ & & + Tx' \\ & & + U \end{array}\right\} = 0 \quad (3)$$

which is of the required form. The roots of equation (2) are equal to those of the reduced equation, each diminished by x', because we have $y = x - x'$.

Denoting the coefficients of the terms of equation (3), after the first, by P', Q', &c., it becomes,

$$y^m + P'y^{m-1} + Q'y^{m-2} \ldots + T'y' + U' = 0 \quad \ldots \quad (4)$$

The operation of finding P', Q', &c., as above indicated, is somewhat tedious, but these coefficients may be obtained by a simpler process, which will now be explained.

Equation (4) was deduced from equation (1) by making, $x = y + x'$; hence, if we make $y = x - x'$ in equation (4), its first member will be identical with the first member of equation (1), and we shall have,

$$(x - x')^m + P'(x - x')^{m-1} + \ldots + S'(x - x')^2 + T'(x - x')$$
$$+ U' = x^m + Px^{m-1} + \ldots + Sx^2 + Tx + U \quad \ldots \quad (5)$$

If we divide the first member of equation (5) by $x - x'$ shall have the quotient

$$(x - x')^{m-1} + P'(x - x')^{m-2} + \ldots + S'(x - x') + T', \quad . \quad (6)$$

with a remainder U′; hence, if we divide the second member by the same quantity, we must get the same quotient and the same remainder; therefore,

If the first member of the given equation be divided by the unknown quantity minus the number that expresses the difference between the roots, the remainder will be the absolute term of the transformed equation.

Again, if we divide the quotient (6), by $x - x'$ we shall find the quotient

$$(x - x')^{m-2} + P'(x - x')^{m-3} + \ldots + S',$$

with the remainder T′, which is the coefficient of the last term but one of the transformed equation. If the new quotient be divided by $x - x'$, the next remainder will be the coefficient of the third term from the last, and so on, till all the coefficients have been found.

EXAMPLES.

1. Transform the equation

$$5x^4 - 12x^3 + 3x^2 + 4x - 5 = 0$$

into another, each of whose roots shall be less than those of the given equation by 2.

FIRST OPERATION.

$$\begin{array}{l||l}
5x^4 - 12x^3 + 3x^2 + 4x - 5 & \underline{\;x - 2\qquad\qquad\quad} \\
\underline{5x^4 - 10x^3} & 5x^3 - 2x^2 - x + 2 \\
\qquad - 2x^3 + 3x^2 & \\
\qquad \underline{- 2x^3 + 4x^2} & \\
\qquad\qquad\qquad - x^2 + 4x & \\
\qquad\qquad\qquad \underline{- x^2 + 2x} & \\
\qquad\qquad\qquad\qquad\quad 2x - 5 & \\
\qquad\qquad\qquad\qquad\quad \underline{2x - 4} & \\
\qquad\qquad\qquad\qquad\quad\; - 1, \text{ 1st remainder.} &
\end{array}$$

SECOND OPERATION.

$$\begin{array}{rrrr|l}
5x^3 & -\ 2x^2 & -\ x & +\ 2 & \underline{x\ -2\qquad} \\
\underline{5x^3} & \underline{-\ 10x^2} & & & 5x^2+8x+15 \\
 & 8x^2 & -\ x & & \\
 & \underline{8x^2} & \underline{-\ 16x} & & \\
 & & 15x & +\ 2 & \\
 & & \underline{15x} & \underline{-\ 30} & \\
 & & & 32, & \text{2d remainder.}
\end{array}$$

THIRD OPERATION.

$$\begin{array}{rrr|l}
5x^2 & +\ 8x & +\ 15 & \underline{x-\ 2\quad} \\
\underline{5x^2} & \underline{-\ 10x} & & 5x+18 \\
 & 18x & +\ 15 & \\
 & \underline{18x} & \underline{-\ 36} & \\
 & & 51, & \text{3d remainder.}
\end{array}$$

FOURTH OPERATION.

$$\begin{array}{rr|l}
5x & +\ 18 & \underline{x-2} \\
\underline{5x} & \underline{-\ 10} & 5 \\
 & 28, & \text{4th remainder.}
\end{array}$$

Hence, the transformed equation is

$$5y^4+28y^3+51y^2+32y-1=0.$$

The operation may be still more abridged by the operation of *Synthetical Division.*

Synthetical Division.

265. Synthetical Division is an abridged method of dividing by detached coefficients. If we take the example of the last article, omitting the literal parts, we have the following

OPERATION.

$$\begin{array}{rrrrr|l}
5 & -12 & +3 & +4 & -5 & \underline{1-\ 2} \\
\underline{5} & \underline{-10} & & & & 5-\ 2-1+2 \\
 & -\ 2 & +3 & & & \\
 & \underline{-\ 2} & \underline{+4} & & & \\
 & & -1 & +4 & & \\
 & & \underline{-1} & \underline{+2} & & \\
 & & & 2 & -5 & \\
 & & & \underline{2} & \underline{-4} & \\
 & & & & -1; &
\end{array}
\qquad
\begin{array}{rrrr|l}
5 & -\ 2 & -1 & +2 & \underline{1-\ 2} \\
\underline{5} & \underline{-10} & & & 5+\ 8+15 \\
 & 8 & -\ 1 & & \\
 & \underline{8} & \underline{-16} & & \\
 & & 15 & +\ 2 & \\
 & & \underline{15} & \underline{-30} & \\
 & & & 32; &
\end{array}
\qquad
\begin{array}{rrr|l}
5 & +\ 8 & +15 & \underline{1-\ 2} \\
\underline{5} & \underline{-10} & & 5+18 \\
 & 18 & +15 & \\
 & \underline{18} & \underline{-36} & \\
 & & 51; &
\end{array}
\qquad
\begin{array}{rr|l}
5 & +18 & \underline{1-2} \\
\underline{5} & \underline{-10} & 5 \\
 & 28; &
\end{array}$$

Here the first remainder is -1; the first quotient might be found by annexing to the coefficients the successive powers of x, from the third down; but for the present purpose, this quotient is not needed. Dividing as before, using the detached coefficients of the first quotient, we find the second remainder 32, and so on. If we examine the operation, we see that the first term in each quotient is 5, the same as the first term in the dividend; the second term is found by multiplying the first by -2, and *subtracting* the result from the second term of the dividend. The same result would have been found by changing the sign of the second term of the divisor, multiplying this result into the first term of the dividend, and then *adding* the product to the second term of the dividend; the third term of the quotient is found by multiplying the second term by the second term of the divisor, with its sign changed, and adding the result to the third term of the dividend; and so on till the last term of the dividend has been used; the last result is the remainder required.

$$\begin{array}{lrrrrrl}
 & 5 & -12 & +3 & +4 & -5 & \lfloor\underline{+2} \\
 & & +10 & -4 & -2 & +4 & \\
\cline{2-6}
\text{1st quotient,} & 5 & -2 & -1 & +2, & -1; & \text{1st rem.} \\
 & & +10 & +16 & +30 & & \\
\cline{2-5}
\text{2d quotient,} & 5 & +8 & +15, & +32; & \text{2d rem.} & \\
 & & 10 & +36 & & & \\
\cline{2-4}
\text{3d quotient,} & 5 & +18, & +51; & \text{3d rem.} & & \\
 & & +10 & & & & \\
\cline{2-3}
\text{4th quotient,} & 5, & +28, & \text{4th rem.} & & &
\end{array}$$

The 4th quotient is the 5th remainder.

Here we have dropped the first term of the divisor and changed the sign of the second term. The first term of the first quotient is 5; multiplying this by 2, and adding to -12, we find -2; multiplying this by 2, and adding to $+3$, we find -1; multiplying this by 2, and adding to

+ 4, we find + 2; multiplying this by 2, and adding to −5, we find the first remainder equal to − 1, which we point off by a comma. In like manner, the successive quotients and remainders are found; beginning at the last remainder, and writing out the transformed equation, we have,

$$5y^4 + 28y^3 + 51y^2 + 32y - 1 = 0.$$

EXAMPLES.

1. Find an equation whose roots are less by 1 than those of the equation

$$x^3 - 7x + 7 = 0.$$

OPERATION.

$$\begin{array}{lccccl}
 & 1 & +0 & -7 & +7 & \big\lfloor \underline{+1} \\
 & & +1 & +1 & -6 & \\ \cline{2-5}
\text{1st quotient,} & 1 & +1 & -6, & +1; & \text{1st rem.} \\
 & & +1 & +2 & & \\ \cline{2-4}
\text{2d quotient,} & 1 & +2, & -4; & \text{2d rem.} & \\
 & & 1 & & & \\ \cline{2-3}
\text{3d quotient,} & 1, & +3; & \text{3d rem.} & &
\end{array}$$

Hence, the required equation is,

$$y^3 + 3y^2 - 4y + 1 = 0.$$

In this case, the term containing x^2 in the given equation is wanting; in all such cases, the places of the missing terms are filled by 0's.

2. Find an equation whose roots are less by 1.7 than those of the equation,

$$x^3 - 2x^2 + 3x - 4 = 0.$$

OPERATION.

$$\begin{array}{lllll}
1 & -\ 2 & +\ 3 & -\ 4 & \big|\,\underline{1.7} \\
 & 1.7 & -\ .51 & +\ 4.233 & \\ \cline{1-4}
1 & -\ 0.3 & +\ 2.49, & .233 & \\
 & +\ 1.7 & +\ 2.38 & & \\ \cline{1-3}
1 & +\ 1.4, & 4.87 & & \\
 & +\ 1.7 & & & \\ \cline{1-2}
1, & +\ 3.1 & & &
\end{array}$$

Hence, the required equation is,

$$y^3 + 3.1y^2 + 4.87y + .233 = 0.$$

3. Find an equation whose roots are greater by 3 than those of the equation,

$$x^4 + 13x^3 + x^2 - 11 = 0.$$

OPERATION.

$$\begin{array}{llllll}
1 & +\ 13 & +\ 1 & +\ 0 & -\ 11 & \big|\,\underline{-\ 3} \\
 & -\ 3 & -\ 30 & +\ 87 & -\ 261 & \\ \cline{1-5}
1 & +\ 10 & -\ 29 & +\ 87, & -\ 272 & \\
 & -\ 3 & -\ 21 & +\ 150 & & \\ \cline{1-4}
1 & +\ 7 & -\ 50, & +\ 237 & & \\
 & -\ 3 & -\ 12 & & & \\ \cline{1-3}
1 & +\ 4, & -\ 62 & & & \\
 & -\ 3 & & & & \\ \cline{1-2}
1, & +\ 1 & & & &
\end{array}$$

Hence, the required equation is,

$$y^4 + y^3 - 62y^2 + 237y - 272 = 0.$$

4. Find the equation whose roots are less by 3 than the roots of the equation

$$x^4 - 3x^3 - 15x^2 + 49x - 12 = 0.$$

Ans. $y^4 + 9y^3 + 12y^2 - 14y = 0$.

5. Find the equation whose roots are less by 10 than the roots of the equation

$$x^4 + 2x^3 + 3x^2 + 4x - 12340 = 0.$$

Ans. $y^4 + 42y^3 + 663y^2 + 4664y = 0.$

6. Find the equation whose roots are less by 2 than the roots of the equation

$$x^5 + 2x^3 - 6x^2 - 10x = 0.$$

Ans. $y^5 + 10y^4 + 42y^3 + 86y^2 + 70y + 4 = 0.$

Disappearance of the Second Term.

266. We see from equation (3), Article **264**, that the second term of the transformed equation will be wanting when

$$mx' + \mathrm{P} = 0, \quad \text{or when} \quad x' = -\frac{\mathrm{P}}{m};$$

but in this case we have in the transformed equation,

$$y = x - x' = x + \frac{\mathrm{P}}{m};$$

that is, in order to find a transformed equation whose second term is wanting, *we find an equation whose roots are greater than those of the given equation by the quotient of the coefficient of the second term by the exponent of the unknown quantity in the first term.*

EXAMPLES.

Transform the following equations so that the second terms shall be wanting in the resulting equations:

1. $x^4 - 12x^3 + 17x^2 - 9x + 7 = 0.$

Here we make the roots of the resulting equation greater than those of the given equation by $-\frac{\mathrm{P}}{4}$ or -3; that is, we make them less than those of the given equation by $+3$. Hence,

$$\begin{array}{rrrrrl}
1 & -12 & +17 & -9 & +7 & \underline{|+3} \\
 & +3 & -27 & -30 & -117 & \\ \hline
1 & -9 & -10 & -39, & -110 & \\
 & +3 & -18 & -84 & & \\ \hline
1 & -6 & -28, & -123 & & \\
 & +3 & -9 & & & \\ \hline
1 & -3, & -37 & & & \\
 & +3 & & & & \\ \hline
1, & -0 & & & &
\end{array}$$

$\therefore$ *Ans.* $y^4 - 37y^2 - 123y - 110 = 0.$

2. $x^5 - 10x^4 + 7x^3 + 4x - 9 = 0.$
   Ans. $y^5 - 33y^3 - 118y^2 - 152y - 73 = 0.$

3. $x^3 - 6x^2 + 7x - 10 = 0.$ *Ans.* $y^3 - 5y - 12 = 0.$

4. $x^3 + 9x^2 - x + 4 = 0.$ *Ans.* $y^3 - 28y + 61 = 0.$

5. $x^4 - 8x^3 + 7x^2 + 3x + 4 = 0.$
   Ans. $y^4 - 17y^2 - 33y - 10 = 0.$

If we suppose

$$m \,.\, \frac{m-1}{2}x'^2 + (m-1)Px' + Q = 0$$

in equation (3), Article **264**, the resulting equation would want its third term. But by solving the above equation we find two values of x', each of which will fulfill the required condition.

The preceding transformations are used in the further discussion and solution of the higher equations.

III. DERIVED POLYNOMIALS.

267. If we assume the reduced equation

$$x^m + Px^{m-1} + Qx^{m-2} + \ldots + Sx^2 + Tx + U = 0 \qquad (1)$$

and in it make $x = u + x'$, we have,

$$(u + x')^m + P(u + x')^{m-1} + \ldots + S(u + x')^2 + T(u + x') + U = 0 \qquad (2)$$

If now we develop the different powers of the binomials and arrange the results according to the ascending powers of u, we shall have,

$$\left.\begin{array}{l|l|l|l}
x'^m & +mx'^{m-1} & u + m\dfrac{m-1}{1.2}x'^{m-2} & u^2 + \ldots u^m \\
+Px'^{m-1} & +(m-1)Px'^{m-2} & +(m-1)\dfrac{m-2}{1.2}Px'^{m-3} & \\
+Qx'^{m-2} & +(m-2)Qx'^{m-3} & +(m-2)\dfrac{m-3}{1.2}Qx'^{m-4} & \\
+\ldots & +\ldots & +\ldots & \\
+Tx' & +T & & \\
+U & & &
\end{array}\right\} = 0$$

Examining the coefficients of the different powers of u, we see that the coefficient of u^0, is what the first member of the given equation becomes when x' is substituted in place of x; we shall denote this expression by X′.

The coefficient of u^1 is formed from the preceding term X′, by multiplying each term of X′ by the exponent of x' in that term, and then diminishing this exponent by 1; we shall denote this coefficient by Y′.

The coefficient of u^2 is formed from Y′, by multiplying each term of Y′ by the exponent of x' in that term, dividing the product by 2, and then diminishing this exponent by 1. Representing this coefficient by $\frac{Z'}{2}$, we see that Z′ *is formed from* Y′ *in the same manner that* Y′ *is formed from* X′.

In general, the coefficient of any power of u, in the bove transformed equation, may be found from the preced- coefficient in the following manner, viz.:

Multiply each term of the preceding coefficient by the exponent of x' in that term, and diminish the exponent of x' by 1; *then divide the algebraic sum of these expressions by the number of preceding coefficients.*

The law by which the coefficients,

$$X', \quad Y', \quad \frac{Z'}{1 . 2}, \quad \frac{V'}{1 . 2 . 3},$$

are derived from each other, is the same as that which governs the formation of the numerical coefficients in the binomial formula.

The expressions, Y′, Z′, V′, W′, &c., are called *successive derived polynomials of* X′, because each is derived from the preceding one by the same law that Y′ is derived from X′.

Generally, any polynomial that is derived from another by the law just explained, is called a *derived polynomial.*

Recollect that X′ is what the given polynomial becomes when x' is substituted for x.

Y′ is called the *first-derived* polynomial;
Z′ is called the *second-derived* polynomial;
V′ is called the *third-derived* polynomial;
&c., &c.

We should also remember that, if we make $u = 0$, we shall have $x' = x$, whence X′ will become the given polynomial, from which the derived polynomials will then be obtained.

EXAMPLES.

1. Find the successive derived polynomials of

$$3x^4 + 6x^3 - 3x^2 + 2x + 1.$$

Calling the given polynomial X′, and denoting the successive derived polynomials by Y′, Z′, V′, and W′, we have from the law just given,

$$\begin{aligned} X' &= 3x^4 + 6x^3 - 3x^2 + 2x + 1 \\ Y' &= 12x^3 + 18x^2 - 6x + 2 \\ Z' &= 36x^2 + 36x - 6 \\ V' &= 72x + 36 \\ W' &= 72 \end{aligned}$$

2. Find the derived polynomials of

$$x^4 - 12x^3 + 17x^2 - 9x + 7.$$

$$\begin{aligned} \textit{Ans.}\quad X' &= x^4 - 12x^3 + 17x^2 - 9x + 7 \\ Y' &= 4x^3 - 36x^2 + 34x - 9 \\ Z' &= 12x^2 - 72x + 34 \\ V' &= 24x - 72 \\ W' &= 24 \end{aligned}$$

Properties of Derived Polynomials.

268. We will now develop some of the properties of derived polynomials.

Let $\quad x^m + Px^{m-1} + Qx^{m-2} \ldots Tx + U = 0$

be a given equation, and a, b, c, d, &c., its m roots. We shall have (Art. **256**)

$$x^m + Px^{m-1} + Qx^{m-2} \ldots = (x-a)(x-b)(x-c) \ldots (x-l).$$

Making

$$x = x' + u,$$

or omitting the accents, and substituting $x + u$ for x, and we have

$$(x+u)^m + P(x+u)^{m-1} + \ldots = (x+u-a)(x+u-b) \ldots ;$$

or, changing the order of x and u, in the second member, and regarding $x - a$, $x - b \ldots$ each as a single quantity,

$$(x+u)^m + P(x+u)^{m-1} \ldots = (u + \overline{x-a})(u + \overline{x-b}) \ldots (u + \overline{x-l}).$$

Performing the operations indicated in the two members, we shall, by the preceding article, obtain for the first member,

$$X + Yu + \frac{Z}{2}u^2 + \ldots u^m;$$

X being the first member of the proposed equation, and Y, Z, &c., the derived polynomials of this member.

With respect to the second member, it follows from Art. **264**:

1°. That the term involving u^0, or the last term, is equal to the product $(x-a)(x-b)\ldots(x-l)$ of the factors of the proposed equation.

2°. The coefficient of u is equal to the sum of the products of these m factors, taken $m-1$ and $m-1$.

3°. The coefficient of u^2 is equal to the sum of the products of these m factors, taken $m-2$ and $m-2$; and so on.

Moreover, since the two members of the last equation are identical, the coefficients of the same powers of u in the two members are equal. Hence,

$$X = (x-a)(x-b)(x-c)\ldots(x-l),$$

which was already shown.

Hence, Y, or the first derived polynomial, *is equal to the sum of the products of the m factors of the first degree in the proposed equation, taken $m-1$ and $m-1$; or equal to the algebraic sum of all the quotients that can be obtained by dividing X by each of the m factors of the first degree in the proposed equation; that is,*

$$Y = \frac{X}{x-a} + \frac{X}{x-b} + \frac{X}{x-c} + \ldots \frac{X}{x-l}.$$

Also, $\frac{Z}{2}$, that is, the second derived polynomial, divided by 2, *is equal to the sum of the products of the m factors of the first member of the proposed equation, taken m — 2 and m — 2; or equal to the sum of the quotients obtained by dividing X by each of the different factors of the second degree; that is,*

$$\frac{Z}{2} = \frac{X}{(x-a)(x-b)} + \frac{X}{(x-a)(x-c)} \dots \frac{X}{(x-k)(x-l)},$$

and so on.

Of Equal Roots.

269. An equation is said to have equal roots when its first member has equal factors of the first degree with respect to the unknown quantity. When this is the case, the derived polynomial, which is the sum of the products of the m factors taken $m-1$ and $m-1$, contains a factor in its different parts, which is two or more times a factor of the first member of the proposed equation (Art. **258**); hence,

There must be a common divisor between the first member of the proposed equation, and its first derived polynomial.

It remains to ascertain the composition of the greatest common divisor.

270. *Having given an equation, it is required to discover whether it has equal roots, and to determine these roots when possible.*

Let us make

$$X = x^m + Px^{m-1} + Qx^{m-2} + \dots + Tx + U = 0,$$

and suppose that the second member contains n factors equal to $x-a$, n' factors equal to $x-b$, n'' factors equal to $x-c \dots$, and also, the simple factors $x-p$, $x-q$, $x-r\ldots$; we shall then have

$$X=(x-a)^{n}(x-b)^{n'}(x-c)^{n''}\ldots(x-p)(x-q)(x-r)\ldots(1)$$

We have seen that Y, or the derived polynomial of X, is *the sum of the quotients obtained by dividing* X *by each of the m factors of the first degree in the proposed equation* (Art. 268).

Now, since X contains n factors equal to $x-a$, we have n partial quotients equal to $\frac{X}{x-a}$; and the same reasoning applies to each of the repeated factors, $x-b$, $x-c\ldots\ldots$ Moreover, we can form but one quotient for each simple factor, which is of the form,

$$\frac{X}{x-p},\quad \frac{X}{x-q},\quad \frac{X}{x-r}\ldots\ldots$$

therefore, the first derived polynomial is of the form,

$$Y=\frac{nX}{x-a}+\frac{n'X}{x-b}+\frac{n''X}{x-c}+\ldots+\frac{X}{x-p}+\frac{X}{x-q}+\frac{X}{x-r}+\ (2)$$

By examining the form of the value of X in equation (1), it is plain that

$$(x-a)^{n-1},\quad (x-b)^{n'-1},\quad (x-c)^{n''-1}\ldots$$

are factors common to all the terms of the polynomial Y; hence the product,

$$(x-a)^{n-1}\times(x-b)^{n'-1}\times(x-c)^{n''-1}\ldots$$

is a divisor of Y. Moreover, it is evident that it will also divide X: it is therefore a common divisor of X and Y; and it is their greatest common divisor.

For, the prime factors of X, are $x-a$, $x-b$, $x-c\ldots$, and $x-p$, $x-q$, $x-r\ldots$; now, $x-p$, $x-q$, $x-r$, cannot divide Y, since some one of them will be wanting in some of the parts of Y, while it will be a factor of the other parts.

Hence, the greatest common divisor of X and Y, is

$$D = (x-a)^{n-1}(x-b)^{n'-1}(x-c)^{n''-1}\ldots;\ \text{that is,}$$

The greatest common divisor is composed of the product of those factors which enter two or more times in the given equation, each raised to a power whose exponent is less by 1 than in the primitive equation.

271. From the above, we deduce the following method for finding the equal roots of the equation,

$$X = 0.$$

1°. *Form* Y, *or the derived polynomial of* X; *then seek for the greatest common divisor between* X *and* Y.

2°. *If one cannot be obtained, the equation has no equal roots.*

If we find a common divisor D, and it is of the first degree, or of the form $x - h$, make $x - h = 0$, whence $x = h$.

3°. *We then conclude, that the equation has two roots equal to h, and has but one species of equal roots, from which it may be freed by dividing* X *by* $(x-h)^2$.

If D is of the second degree with reference to x, *solve the equation* $D = 0$. There may be two cases; the two roots may be equal, or they may be unequal.

1°. When $D = (x-h)^2$, *the equation has three roots equal to h, and has but one species of equal roots,* from which it can be freed by dividing X by $(x-h)^3$.

2°. When D is of the form $(x-h)(x-h')$, the proposed equation *has two roots equal to h,* and *two equal to*

h', from which it may be freed by dividing X by $(x-h)^2$ $(x-h')^2$.

Suppose D to be of any degree whatever; *it is necessary,* in order to know the species of equal roots, and the number of roots of each species, *to solve the equation,*

$$D=0.$$

Then, every single root of the equation $D=0$ *will be twice a root of the given equation; and every quantity that is twice a root, will be three times a root of the given equation; and so on.*

To find the simple roots of

$$X=0,$$

we free the equation of its equal roots; the resulting equation, $X'=0$, will then make known the simple roots.

EXAMPLES.

1. Determine whether the equation,

$$2x^4-12x^3+19x^2-6x+9=0,$$

has any equal roots.

We have for the first derived polynomial of the first member,

$$8x^3-36x^2+38x-6.$$

Now, seeking for the greatest common divisor of the first member and its first derived polynomial, we have,

$$D=x-3=0, \quad \text{whence} \quad x=3:$$

hence, the given equation has *two* roots equal to 3.

Dividing its first member by $(x-3)^2$, we obtain

$$2x^2+1=0; \quad \text{whence,} \quad x=\pm\frac{1}{2}\sqrt{-2}.$$

The equation, therefore, is completely solved, and its roots are

$$3, \quad 3, \quad +\frac{1}{2}\sqrt{-2} \text{ and } -\frac{1}{2}\sqrt{-2}.$$

2. For a second example, take

$$x^5 - 2x^4 + 3x^3 - 7x^2 + 8x - 3 = 0.$$

The first derived polynomial is

$$5x^4 - 8x^3 + 9x^2 - 14x + 8;$$

and the common divisor,

$$x^2 - 2x + 1 = (x-1)^2;$$

hence, the proposed equation has *three* roots equal to 1.

Dividing its first member by

$$(x-1)^3 = x^3 - 3x^2 + 3x - 1,$$

the quotient is

$$x^2 + x + 3 = 0; \quad \text{whence,} \quad x = \frac{-1 \pm \sqrt{-11}}{2};$$

hence, the equation is completely solved.

3. For a third example, take the equation

$$x^7 + 5x^6 + 6x^5 - 6x^4 - 15x^3 - 3x^2 + 8x + 4 = 0.$$

The first derived polynomial is

$$7x^6 + 30x^5 + 30x^4 - 24x^3 - 45x^2 - 6x + 8;$$

and the common divisor is

$$x^4 + 3x^3 + x^2 - 3x - 2.$$

The equation,

$$x^4 + 3x^3 + x^2 - 3x - 2 = 0,$$

cannot be solved directly, but by applying the method of equal roots to it, that is, by seeking for a common divisor between its first member and its derived polynomial,

$$4x^3 + 9x^2 + 2x - 3:$$

we find a common divisor, $x + 1$; which proves that the *square* of $x + 1$ is a factor of

$$x^4 + 3x^3 + x^2 - 3x - 2,$$

and the *cube* of $x + 1$, a factor of the first member of the given equation.

Dividing

$$x^4 + 3x^3 + x^2 - 3x - 2 \quad \text{by} \quad (x + 1)^2 = x^2 + 2x + 1,$$

we have $x^2 + x - 2$, which being placed equal to zero, gives the two roots $x = 1$, $x = -2$, or the two factors, $x - 1$ and $x + 2$. Hence, we have

$$x^4 + 3x^3 + x^2 - 3x - 2 = (x + 1)^2 (x - 1)(x + 2).$$

Therefore, the first member of the proposed equation is equal to

$$(x + 1)^3 (x - 1)^2 (x + 2)^2;$$

that is, the proposed equation has *three* roots equal to -1, *two* equal to $+1$, and *two* equal to -2.

4. What is the product of the equal factors in the first member of the equation

$$x^7 - 7x^6 + 10x^5 + 22x^4 - 43x^3 - 35x^2 + 48x + 36 = 0?$$

Ans. $(x - 2)^2 (x - 3)^2 (x + 1)^3$.

5. What is the product of the equal factors in the first member of the equation,

$$x^7 - 3x^6 + 9x^5 - 19x^4 + 27x^3 - 33x^2 + 27x - 9 = 0?$$

Ans. $(x - 1)^3 (x^2 + 3)^2$.

IV. GENERAL ELIMINATION.

272. We have already explained the methods of eliminating one unknown quantity from two equations, when these equations are of the first degree with respect to the unknown quantities.

When the equations are of a higher degree than the first, those methods are not generally applicable. In this case, *the method of the greatest common divisor* is considered the best, and it is this method that we now propose to explain.

One quantity is said to be a *function* of another when it depends on that quantity for its value; that is, when the quantities are so connected, that the value of the latter cannot be changed without producing a corresponding change in the former.

273. Final Equation.—If two equations, containing two unknown quantities, be combined, so as to produce a single equation containing but one unknown quantity, the resulting equation is called a *final equation;* and the roots of this equation are called *compatible values* of the unknown quantity that enters it.

Let us assume the equations,

$$P = 0 \quad \text{and} \quad Q = 0,$$

in which P and Q are functions of x and y of any degree whatever; it is required to combine these equations so as to eliminate one of the unknown quantities.

If we suppose the final equation containing y to be found, and that $y = a$ is a root of this equation, it is plain that this value of y, in connection with some value of x, will satisfy both equations.

If we substitute this value of y in both equations, there will result two equations containing only x, and these equations will have at least one root in common, and consequently, their first members will have a common divisor involving x (Art. **256**).

This common divisor will be of the first, or of a higher degree with respect to x, according as the particular value of $y=a$ corresponds to one or more values of x.

Conversely, *every value of y which*, being substituted in the two equations, gives a common divisor involving x, *is necessarily a compatible value*, for it then satisfies the two equations at the same time with the value or values of x found from this common divisor when put equal to 0.

274. Before *the substitution, the first members of the equations cannot have a common divisor* that is a function of one or both of the unknown quantities.

For, suppose the equations

$$P=0 \quad \text{and} \quad Q=0,$$

are of the form

$$P' \times R=0 \quad \text{and} \quad Q' \times R=0,$$

R being a function of both x and y.

Placing $R=0$, we obtain a single equation containing two unknown quantities, which can be satisfied by an *infinite number of sets of values.* Moreover, every system that renders R equal to 0, at the same time causes $P' \times R$ and $Q' \times R$ to become 0, and consequently, satisfies the equations

$$P=0 \quad \text{and} \quad Q=0.$$

Thus, the hypothesis of a common divisor of the two polynomials P and Q, containing x and y, brings with it, as a consequence, that the proposed equations are indeterminate. Therefore, if there is a common divisor, involving x and y, between the two polynomials P and Q, the proposed equations are *indeterminate;* that is, they may be satisfied by an infinite number of sets of values of x and y.

Again, suppose that R is a function of x only.

Placing $R = 0$, and solving the equation with reference to x, we obtain one or more values for this unknown quantity.

Each of these values, substituted in the equations

$$P' \times R = 0 \quad \text{and} \quad Q' \times R = 0,$$

will satisfy them, whatever value we attribute to y, since these values of x reduce R to 0, independently of y. Therefore, in this case, the proposed equations admit of a *finite number of values* for x, but of an infinite number of values for y; therefore there cannot be a finite equation in y.

Hence, when the equations

$$P = 0, \quad Q = 0,$$

are determinate, that is, when they admit only of a *limited number* of sets of values for x and y, their first members cannot have *a common divisor that is a function of these unknown quantities,* until a particular substitution has been made for one of them.

275. From this we deduce a process for obtaining the *final equation* involving y.

The characteristic property of a compatible value of y is, that when substituted in the first members of the two equations, it gives them a common divisor in x, that they had not before; hence, if to the two proposed polynomials, arranged with reference to x, we apply the process for finding the greatest common divisor, we shall generally not find one. But, by continuing the operation properly, we shall arrive at a remainder independent of x, but containing y; this, placed equal to 0, gives the *final equation.*

For, any value of y found from this equation, reduces the last remainder in the operation for finding the common divisor to 0; hence, if it be substituted in the preceding remainder, it will render this remainder a common divisor of P and Q. Therefore, each root of the equation thus formed, is a compatible value of y.

276. Admitting that the final equation may be completely solved, which would give all the compatible values, it would then be necessary to obtain the corresponding values of x. Now, it is sufficient for this, to substitute the different values of y in the remainder preceding the last, put the resulting polynomials equal to 0, and find from them the values of x; for these polynomials are the divisors involving x, that become common to A and B.

But as the final equation is generally of a degree superior to the second, we cannot here explain the methods of finding the values of y. Indeed, our design was to show that, *two equations of any degree being given, we can find another equation, containing only one of the unknown quantities that enter the proposed equations.*

EXAMPLES.

1. Given the equations

$$\begin{aligned} x^2 + xy + y^2 - 1 &= 0, \\ x^3 + y^3 &= 0, \end{aligned}$$

to find the final equation in y.

FIRST OPERATION.

$$\begin{array}{r|l} x^3 + y^3 \qquad\qquad\qquad & x^2 + xy + y^2 - 1 \\ \cline{2-2} x^3 + yx^2 + (y^2 - 1)x & x - y \\ \cline{1-1} -yx^2 - (y^2 - 1)x + y^3 & \\ -yx^2 - y^2x - y^3 \quad + y & \\ \cline{1-1} x + 2y^3 - y, & \text{1st remainder.} \end{array}$$

SECOND OPERATION.

$$\begin{array}{l|l}
x^2 + yx \quad + y^2 - 1 & \underline{x + 2y^3 - y} \\
\underline{x^2 + (2y^3 - y)x} & x - (2y^3 - 2y) \\
-(2y^3 - 2y)x + \ y^2 - 1 & \\
\underline{-(2y^3 - 2y)x - 4y^6 + 6y^4 - 2y^2} & \\
\qquad 4y^6 - 6y^4 + 3y^2 - 1. &
\end{array}$$

Hence, the final equation in y, is

$$4y^6 - 6y^4 + 3y^2 - 1 = 0.$$

It may be observed that x and y enter the primitive equations under the same forms; hence, x may be changed into y, and y into x, without destroying the equality of the members. Therefore,

$$4x^6 - 6x^4 + 3x^2 - 1 = 0$$

is the final equation in x.

2. Find the final equation in y, from the equations

$$\begin{array}{l}
x^3 - 3yx^2 + (3y^2 - y + 1)x - y^3 + y^2 - 2y = 0, \\
x^2 - 2yx + \quad y^2 - y = 0.
\end{array}$$

FIRST OPERATION.

$$\begin{array}{l|l}
x^3 - 3yx^2 + (3y^2 - y + 1)x - y^3 + y^2 - 2y & \underline{x^2 - 2xy + y^2 - y} \\
\underline{x^3 - 2yx^2 + \ (y^2 - y)x} & x - y \\
\quad - \ yx^2 + (2y^2 + 1)x - y^3 + y^2 - 2y & \\
\quad - \ yx^2 + \ 2y^2x \qquad \underline{- y^3 + y^2} & \\
\qquad\qquad\qquad\qquad x - 2y &
\end{array}$$

A second operation gives for the final equation in y, $y^2 - y = 0$, of which the roots are $y = 1$ and $y = 0$. The corresponding values of x are 2 and 0, from which the given equation may be entirely solved.

CHAPTER XI.

SOLUTION OF NUMERICAL EQUATIONS.

I. GENERAL PROPERTIES OF NUMERICAL EQUATIONS.

277. A Numerical Equation is one in which the coefficients of the different powers of the unknown quantity are numbers.

The properties of equations deduced in the last chapter hold true for all kinds of equations, whether *numerical* or *algebraic*. These properties form the basis of the principal methods employed in solving numerical equations, some of which will be now investigated.

Before proceeding to the methods of solution, it will be necessary to establish a few preliminary principles.

278. Principles.—Let us resume the reduced equation

$$x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U = 0 \quad . \quad . \quad (1)$$

and suppose that P, Q, ... T, and U are numerical, either positive or negative.

Let us denote the first member of equation (1) by X, and its successive derived polynomials by Y, Z, V, &c.

If we replace x, in equation (1), by $x + u$, we shall have, for the transformed equation, when arranged according to the ascending powers of u, as explained in Article **271**,

$$X + Yu + \frac{Z}{1.2}u^2 + \frac{V}{1.2.3}u^3 + \ldots + u^m = 0 \quad . \quad . \quad (2)$$

1° Principle.—*If we substitute a for x, in equation* (1), *and denote what* X *becomes, under that hypothesis, by* A, *and then substitute $a+u$ for x, and denote what* X *becomes under that hypothesis by* A′; *then may u be taken so small as to make the difference between* A′ *and* A *less than any assignable number.*

The value of A′ will be found by making $x=a$ in the coefficients of the different powers of u, in equation (2). Denoting what Y, $\frac{Z}{1.2}$, $\frac{V}{1.2.3}$, &c., become when $x=a$, by B, C, D, &c., we have,

$$A' = A + Bu + Cu^2 + Du^3 + \ldots + u^m \quad . \quad . \quad . \quad (3)$$

Transposing A to the first member, we have,

$$A' - A = Bu + Cu^2 + Du^3 + \ldots + u^m \quad . \quad . \quad (4)$$

To make the difference between A′ and A less than any number N, we must assign a value to u that will satisfy the inequality

$$Bu + Cu^2 + Du^3 + \ldots . u^m < N \quad . \quad . \quad . \quad . \quad (5)$$

Let us take the most unfavorable case that can occur, viz., when every coefficient is positive, and each equal to the greatest, which denote by K. Then, any value of u that will satisfy the inequality

$$K(u + u^2 + u^3 + \ldots . u^m) < N \quad . \quad . \quad . \quad . \quad (6)$$

will evidently satisfy inequality (5).

Now, the expression within the parenthesis is a geometrical progression, whose first term is u, whose last term is u^m, and whose ratio is u; hence (Art. **187**),

$$u+u^2+u^3+ \ldots u^m = \frac{u^{m+1}-u}{u-1} = \frac{u-u^{m+1}}{1-u} = \frac{u}{1-u} \times (1-u^m).$$

Substituting this value in inequality (6), we have,

$$\frac{Ku}{1-u}(1-u^{m}) < N \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we make $u = \frac{N}{N+K}$, the first factor of the first member of inequality (7) reduces to N, and since $\left(\frac{N}{N+K}\right)^{m}$ is less than 1, the second factor is less than 1; consequently, the first member is less than N.

Hence, $u = \frac{N}{N+K}$, and every smaller value of u, will satisfy inequalities (5) and (6), and consequently, make the difference between A′ and A less than N, which was to be shown.

If, in equation (3), we make $u = \frac{A}{A+K}$, in the value of A′, the sum of the terms

$$Bu + Cu^{2} + Du^{3} + \ldots u^{m}$$

will be less than A, from what has just been proved; hence,

In a series of terms arranged according to the ascending powers of an arbitrary quantity, a value may be assigned to that quantity so small, as to make the first term numerically greater than the sum of all the other terms.

2° **Principle.**—*If the two numbers p and q, when substituted for x in the first member of equation* (1), *give results having contrary signs, then will that equation have at least one real root whose value lies between p and q.*

Let p be less than q, that is, nearer in value to $-\infty$, and suppose that p, when substituted for x, reduces X to $+$ R, and that q, when substituted for x, reduces X to $-$ R′.

Let us suppose x to vary, increasing in value from p toward q by infinitesimal increments; that is, by increments so small that the corresponding consecutive values of X shall differ from each other by less than any assignable quantity. Under this hypothesis, we say that both x and X are subjected to the *law of continuity;* that is, if x passes through all intermediate values from p to q, X will pass through all values it can have between the limits R and $-R'$. From the nature of the case, all the values of X, between these limits, must be finite.

Now, a finite quantity subject to the law of continuity cannot change its sign from + to −, or from − to +, without passing through 0; hence, there is some value of x between p and q that will make

$$X = 0,$$

and consequently, this equation has at least one real root lying between p and q.

3° Principle.—*If equation* (1) *has an odd number of real roots lying between p and q, the results obtained by substituting these numbers for x in the first member of equation* (1) *will have contrary signs; if it has an even number of real roots lying between these numbers, the results found by the substitutions will have the same signs.*

Let a, b, c, . . . denote the roots that lie between p and q, and let Y denote the product of all the factors of the first degree, with reference to x, corresponding to the remaining roots of the given equation.

The first member, X, can then be put under the form

$$(x-a)(x-b)(x-c)\ldots \times Y = 0.$$

Now, substituting p and q in place of x, in the first member, we obtain

$$(p-a)(p-b)(p-c)\ldots \times Y'$$
$$(q-a)(q-b)(q-c)\ldots \times Y'',$$

Y' and Y'' representing what Y becomes, when we replace x by p and q. The quantities Y' and Y'', are affected with the same sign; for, if they were not, by the first principle there would be at least one other real root comprised between p and q, which is contrary to the hypothesis.

To determine the signs of the above results more readily, divide the first by the second, and we obtain

$$\frac{(p-a)(p-b)(p-c)\ldots \times Y'}{(q-a)(q-b)(q-c)\ldots \times Y''},$$

which can be written thus,

$$\frac{p-a}{q-a} \times \frac{p-b}{q-b} \times \frac{p-c}{q-c} \times \ldots \frac{Y'}{Y''}.$$

Now, since the root a is comprised between p and q, that is, is greater than one and less than the other, $p-a$ and $q-a$ must have contrary signs; also, $p-b$ and $q-b$ must have contrary signs, and so on.

Hence, the quotients

$$\frac{p-a}{q-a},\ \frac{p-b}{q-b},\ \frac{p-c}{q-c},\ \&c.,$$

are all *negative.*

Moreover, $\frac{Y'}{Y''}$ is essentially positive, since Y' and Y'' are affected with the same sign; therefore, the product

$$\frac{p-a}{q-a} \times \frac{p-b}{q-b} \times \frac{p-c}{q-c} \times \ldots \frac{Y'}{Y''},$$

will be *negative,* when the number of roots, $a, b, c \ldots$, comprehended between p and q, is uneven, and *positive* when the number is even.

Consequently, the two results,

$$(p-a)(p-b)(p-c)\ldots \times Y',$$

and

$$(q-a)(q-b)(q-c)\ldots \times Y'',$$

will have contrary signs when the number of roots comprised between p and q is *uneven,* and the same sign when the number is *even.*

4° **Principle.**—*If equation* (1) *is complete* (*Art.* **255**), *and the signs of the alternate terms be changed, the signs of all the roots of the equation will be changed.*

Take the equation,

$$x^m + Px^{m-1} + Qx^{m-2} \dots + U = 0 \quad . \; . \; . \quad (1)$$

By changing the signs of the alternate terms, we have either,

$$x^m - Px^{m-1} + Qx^{m-2} \dots \pm U = 0 \quad . \; . \; . \quad (8)$$

or,

$$-x^m + Px^{m-1} - Qx^{m-2} \dots \mp U = 0 \quad . \; . \; . \quad (9)$$

But equations (8) and (9) are the same, since the sum of the positive terms of the one is equal to the sum of the negative terms of the other, whatever be the value of x.

Suppose a to be a root of equation (1); then, the substitution of a for x will verify that equation. But the substitution of $-a$ for x, in either of equations (8) or (9), will give the same result as the substitution of $+a$, in equation (1): hence, $-a$ is a root of equation (8), or of equation (9).

We also conclude, that if the signs of *all* the terms be changed, the signs of the roots will not be altered.

II. LIMITS OF THE REAL ROOTS.

279. Superior Limit.—*A superior limit* of the roots of a numerical equation, is a number greater than the greatest root of that equation; that is, a number whose value is nearer to $+\infty$ than any root of the equation.

Let us resume the numerical equation

$$x^m + Px^{m-1} + Qx^{m-2} + \dots + Tx + U = 0 \quad . \quad (1)$$

In the first place, let us find a value of x that will satisfy the inequality,

$$x^m > Px^{m-1} + Qx^{m-2} + \dots + Tx + U \quad . \quad . \quad (2)$$

Let k denote the value of the greatest numerical coefficient, that is, the greatest coefficient without regard to its sign, and substitute it in place of each of the coefficients: inequality (2) will then become

$$x^m > kx^{m-1} + kx^{m-2} + \dots + kx + k \quad . \quad . \quad (3)$$

It is evident that any value of x that will satisfy this inequality will satisfy the preceding one. Dividing both members of inequality (3) by x^m, we have,

$$1 > \frac{k}{x} + \frac{k}{x^2} + \frac{k}{x^3} + \dots + \frac{k}{x^{m-1}} + \frac{k}{x^m} \quad . \quad . \quad . \quad (4)$$

If we make $x = k$, in inequality (4), it is obvious that the inequality will not be satisfied; but if we make $x = k + 1$, we have,

$$1 > \frac{k}{k+1} + \frac{k}{(k+1)^2} + \frac{k}{(k+1)^3} + \dots + \frac{k}{(k+1)^m} \quad (5)$$

The second member of inequality (5) is a geometrical progression, in which the first term is $\frac{k}{k+1}$, the ratio $\frac{k}{k+1}$, and the last term $\frac{k}{(k+1)^m}$. Finding the sum of this progression, by the formula of Article **187**, and reducing, we have,

$$1 > 1 - \frac{1}{(k+1)^m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which is a true inequality. It is obvious, furthermore, that any number greater than $k + 1$ will make the second member of inequality (4), still less than the second member of inequality (6). Consequently, $k+1$ is a superior limit of the roots of equation (1). Hence, to find a superior limit we have the following

RULE.

Find the greatest coefficient of any term (disregarding its sign), and increase its numerical value by 1.

EXAMPLE.

A superior limit of the roots of the equation

$$x^4 + 17x^3 - 20x - 6 = 0$$

is $20+1$ or 21.

280. Ordinary Superior Limit.—A superior limit found by the preceding rule is commonly too great, because the equation usually contains several positive terms.

To find a less limit, let x^{m-n} be the power of x that enters the first negative term of the equation, and let us take the most unfavorable case, viz., that in which the coefficients of all the succeeding terms are negative, and each equal to the greatest negative coefficient in the equation numerically considered. Let this coefficient, disregarding its sign, be denoted by S.

Let us find the conditions that will satisfy the inequality

$$x^m > Sx^{m-n} + Sx^{m-n-1} + \ldots + Sx + S \quad . \quad . \quad (1)$$

Dividing both members by x^m, we have,

$$1 > \frac{S}{x^n} + \frac{S}{x^{n+1}} + \ldots + \frac{S}{x^{m-1}} + \frac{S}{x^m} \quad . \quad . \quad . \quad (2)$$

If we make $x = \sqrt[n]{S} + 1$, or, representing $\sqrt[n]{S}$ by S', which gives, $S = S'^n$ and $x = S' + 1$, inequality (2) becomes

$$\frac{S'^n}{(S'+1)^n} + \frac{S'^n}{(S'+1)^{n+1}} + \ldots + \frac{S'^n}{(S'+1)^{m-1}} + \frac{S'^n}{(S'+1)^m} \, (3)$$

The second member of inequality (3) is a geometrical progression in which the first term is $\frac{S'^n}{(S'+1)^m}$, the ratio $\frac{1}{S'+1}$, and the last term, $\frac{S'^n}{(S'+1)^m}$. Finding the sum of this progression, and reducing as before, we have,

$$1 > \frac{S'^{n-1}}{(S'+1)^{n-1}} - \frac{S'^{n-1}}{(S'+1)^m} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The second member is the difference between two proper fractions; hence, this is a true inequality. Furthermore, any number greater than $S'+1$, when substituted for x in inequality (2), will satisfy it; consequently, $S'+1$ is a superior limit of the roots of equation (1), Article **279**. This limit is called *the ordinary superior limit.* Hence, to find the ordinary superior limit, we have the following

RULE.

Find the greatest coefficient of any negative term (disregarding its sign), extract that root of it whose index is the number of terms that precede the first negative term, and to the result add 1.

EXAMPLES.

Find the ordinary superior limits of the roots of the following equations:

1. $$x^4 + 17x^2 - 20x - 6 = 0.$$

Here the greatest numerical coefficient is 20, and the number of terms that precede the first negative term is 3 (counting the term that is wanting); hence, the required limit is $\sqrt[3]{20}+1$. In whole numbers, the limit is 4.

2. $x^4 - 5x^3 + 37x^2 - 3x + 39 = 0.$

Ans. $\sqrt[n]{S}+1 = \sqrt[1]{5}+1 = 6.$

3. $x^5 + 7x^4 - 12x^3 - 49x^2 + 52x - 13 = 0.$

Ans. $\sqrt{49} + 1 = 8.$

4. $x^4 + 11x^2 - 25x - 67 = 0.$ *Ans.* $\sqrt[3]{67} + 1.$

The least whole number that will satisfy the conditions is 6.

281. Least Superior Limit in Whole Numbers.— In article **279**, a rule was deduced for finding a superior limit; in Article **280** a rule was deduced for finding a less superior limit; it is now proposed to find the *least* superior limit, in whole numbers.

Let $$X = 0$$

be the proposed equation. If in this equation we make $x = x' + u$, x' being arbitrary, we obtain, as in Article **278**,

$$X' + Y'u + \frac{Z'}{2}u^2 + \ldots + u^m = 0 \quad . \quad . \quad . \quad . \quad (1)$$

Let us determine, by trial, a value for x', which substituted in

$$X', \quad Y', \quad \frac{Z'}{2} \ldots,$$

will render all these coefficients positive; this number will be greater than the greatest positive root of the equation

$$X = 0.$$

For, if the coefficients of equation (1) are all positive, no positive value of u can satisfy it; therefore, *all* the real values of u must be negative. But from the equation

$$x = x' + u, \quad \text{we have} \quad u = x - x';$$

and in order that every value of u, corresponding to each of the values of x and x', may be negative, it is necessary that the greatest positive value of x should be less than the value

The value of x', determined by this method, is a superior limit, but it may not be the least superior limit in whole numbers. To test it, let $x'-1$, $x'-2$, $x'-3$, &c., be substituted successively for x in the polynomial X, until one is found that renders it negative; the preceding one is the least limit required.

EXAMPLES.

Find the least superior limit, in whole numbers, of the roots of the following equations:

$$x^4-5x^3-6x^2-19x+7=0.$$

Dropping the dashes, we have,

$$\begin{aligned}
X &= x^4-5x^3-6x^2-19x+7,\\
Y &= 4x^3-15x^2-12x-19,\\
\frac{Z}{2} &= 6x^2-15x-6,\\
\frac{V}{2.3} &= 4x-5.
\end{aligned}$$

We see that all numbers greater than 2 render V positive; all greater than 3 make Z positive; all greater than 5 make Y positive; and all numbers greater than 7 render X positive, but 6 renders X negative. Hence, 7 is the required limit.

2. $x^5-3x^4-8x^3-25x^2+4x-39=0.$ *Ans.* 6.

282. Inferior Limit.—An inferior limit of the roots of a numerical equation is a number algebraically less than the least root; that is, it is a number whose value is nearer to $-\infty$ than any of the roots of the equation.

1°. When all the roots of an equation involving x are positive, the greatest inferior limit will be positive.

To find an inferior limit, make $x = \frac{1}{y}$, and find the transformed equation, $Y = 0$. The greatest value of y in this equation will be the reciprocal of the least value of x in the given equation. Hence, if we find a superior limit of the roots of the equation $Y = 0$, and denote it by L, an inferior limit of the roots of the given equation will be equal to $\frac{1}{L}$.

2°. When some of the roots are negative, the inferior limits will be negative. To find one of them, we make $x = -y$, and find the transformed equation, $Y' = 0$. The greatest root of this equation will be equal to the least root of the given equation, algebraically considered. Hence, if we find a superior limit of the roots of the equation $Y' = 0$, and denote it by L', then an inferior limit of the roots of the given equation, will be equal to $-L'$.

Consequences.

283. The following consequences are deducible from the preceding principles:

1°. *Every equation in which there are no variations in the signs,* that is, *in which all the terms are positive, has all its real roots negative;* for, every positive number substituted for x renders the first member essentially positive.

A *variation* is a change of sign in passing along a succession of quantities; a *permanence* occurs when two successive ones have the same sign. Thus, if the signs of a series of quantities occur in the following order:

$$+ + - + - - +,$$

we say that there are 4 *variations* of sign and 2 permanencies.

2°. *Every complete equation,* having its terms alternately e and negative, has all its real roots positive; for, every negative number substituted for x in the proposed equation renders all the terms positive, if the equation is of an even degree, and all of them negative, if it is of an odd degree. Hence, their sum is not equal to zero in either case.

This principle is also true for *every incomplete equation, in which the substitution of* $-y$ *for* x, *gives an equation having all its terms affected with the same sign.*

3°. Every equation of an odd degree, whose coefficients are real, *has at least one real root affected with a sign contrary to that of its last term.*

For, let

$$x^m + Px^{n-1} + \ldots Tx \pm U = 0,$$

be the proposed equation; and first consider the case in which the last term is *negative.*

By making $x = 0$, the first member becomes $-U$. But by giving a value to x equal to the greatest coefficient plus 1, or $(K + 1)$, the first term x^m will become greater than the arithmetical sum of all the others (Art. **283**); the result of this substitution is therefore *positive;* hence, *there is at least one real root between* 0 *and* $K + 1$, which root is positive, and consequently affected with a sign *contrary* to that of the last term (Art. **283**).

Suppose now the last term *positive.*

Making $x = 0$ in the first member, we obtain $+U$ for the result; but by putting $-(K + 1)$ in place of x, we obtain a *negative* result, since the first term becomes negative by this substitution; hence, the equation has at least one real root between 0 and $-(K + 1)$, which is negative, or *affected with a sign contrary* to that of the last term.

4°. *Every equation of an even degree,* that contains only real coefficients, *and of which the last term is negative, has at least two real roots, one positive and the other negative.*

For, let $-U$ be the last term; making $x=0$, there results $-U$. Now, substitute either $K+1$, or $-(K+1)$, K being the greatest coefficient in the equation. As m is an even number, the first term x^m will remain positive; besides, by these substitutions, it becomes greater than the sum of all the others; therefore, the results obtained by these substitutions are both *positive*, or affected with a sign contrary to that when $x=0$; hence, the equation *has at least two real roots*, one *positive*, and lying between 0 and $K+1$, the other *negative*, and lying between 0 and $-(K+1)$.

5°. *If an equation, involving only real coefficients, contain imaginary roots, the number of such roots must be even.*

For, conceive the first member to have been divided by all the simple factors corresponding to real roots; the coefficients of the *quotient* will be real; *and the quotient must be of an even degree;* for, if it be of an odd degree, by placing it equal to zero, we obtain an equation that would contain at least one real root, in accordance with consequence 3°. Hence, the degree of an equation whose roots are all imaginary is even, and consequently, the number of imaginary roots is even. Furthermore, the sign of the first member is positive for all values of x. For, if we make $x=\infty$, it is obvious that the first member will be equal to $+\infty$; if any value of x, as a, could render the first member *negative*, there would be at least one real root between a and $+\infty$, which is contrary to the hypothesis.

6°. *When the last term of an equation is positive, the number of its real positive roots is even; and when it is negative, the number of such roots is uneven.*

For, first suppose that the last term is $+U$, or *positive.* Since by making $x=0$, there will result $+U$, and by making $x=K+1$, the result will also be positive, it llows that 0 and $K+1$ give two results affected with the same sign, and consequently the number of real roots, if any, comprehended between them, is even.

When the last term is $-U$, then 0 and $K+1$ give two results affected with contrary signs, and consequently, they comprehend either a *single root,* or *an odd number of them.*

The converse of this proposition is evidently true.

III. COMMENSURABLE ROOTS.

284. An equation of the form,

$$x^m + Px^{m-1} + Qx^{m-2} + \ldots Tx + U = 0,$$

can have no commensurable roots that are not whole numbers, if the coefficients are entire.

For, if possible, let one of its roots be an irreducible fraction $\frac{a}{b}$. Substituting this for x in the given equation, we have,

$$\frac{a^m}{b^m} + P\frac{a^{m-1}}{b^{m-1}} + Q\frac{a^{m-2}}{b^{m-2}} + \ldots + T\frac{a}{b} + U = 0;$$

whence, multiplying both members by b^{m-1}, and transposing,

$$\frac{a^m}{b} = -Pa^{m-1} - Qa^{m-2}b - \ldots - Tab^{m-2} - Ub^{m-1}.$$

But the second member of this equation is composed of entire numbers, while the first is essentially fractional, for a and b being prime with respect to each other, a^m and b will also be prime with respect to each other; hence, this equality cannot exist; for, an irreducible fraction cannot be equal to a whole number. Therefore, it is impossible for any irreducible fraction to satisfy the equation.

Now, it has been shown (Art. **261**), that an equation containing fractional coefficients, can be transformed into another in which the coefficients are whole numbers, that of the first term being 1. Hence, *the search for commensurable roots*, either entire or fractional, *can always be reduced to that for entire roots of a transformed equation.*

285. Assume the general equation

$$x^m + Px^{m-1} + Qx^{m-2} \dots + Rx^3 + Sx^2 + Tx + U = 0,$$

and let a be any entire number, positive or negative, that will satisfy it.

Since a is a root, we have the equation

$$a^m + Pa^{m-1} + \dots + Ra^3 + Sa^2 + Ta + U = 0 \quad . \quad (1)$$

Now replace a by all the entire numbers, positive and negative, between 1 and the limit $+L$, and between -1 and $-L'$: those that satisfy the above equality will be roots of the equation. But these trials being long and troublesome, we will deduce from equation (1), other conditions equivalent to this, and more easily applied.

Transposing all the terms except the last, in equation (1), and dividing by a, we have,

$$\frac{U}{a} = -a^{m-1} - Pa^{m-2} - \dots - Ra^2 - Sa - T \quad . \quad (2)$$

Now, the second member of this equation is entire; hence, $\frac{U}{a}$ must be entire; therefore, *the entire roots of the equation are comprised among the divisors of the last term.*

Transposing $-T$ in equation (2), dividing by a, and making

$$\frac{U}{a} + T = T', \text{ we have,}$$

$$\frac{T'}{a} = -a^{m-2} - Pa^{m-3} \dots - Ra - S \quad . \quad . \quad . \quad (3)$$

The second member of this equation being entire, $\frac{T'}{a}$, that is, *the quotient of*

$$\frac{U}{a}+T \text{ by } a,$$

is entire.

Transposing $-S$, and dividing by a, we have, and supposing

$$\frac{T'}{a}+S=S',$$

$$\frac{S'}{a}=-a^{m-3}-Pa^{m-4}-\ldots-R \quad . \quad . \quad . \quad (4)$$

The second member of this equation being entire, $\frac{S'}{a}$, that is, *the quotient of*

$$\frac{T'}{a}+S \text{ by } a,$$

is entire.

By continuing to transpose the terms of the second member into the first, we shall, after $m-1$ transformations, obtain an equation of the form,

$$\frac{Q'}{a}=-a-P.$$

Then, transposing the term $-P$, dividing by a, and making

$$\frac{Q'}{a}+P=P', \text{ we have } \frac{P'}{a}=-1, \text{ or } \frac{P'}{a}+1=0.$$

This equation, that results from the continued transformations of equation (1), expresses the *last condition which it is requisite for* the number a to fulfil, that it may be a root of the equation.

From what precedes we conclude that, when a number a, positive or negative, is a root of the given equation, *the quotient of the last term by* a, *is a whole number.*

Adding to this quotient the coefficient of x^1, *the sum is divisible by* a.

Adding the coefficient of x^2 to this last quotient, and again dividing by a, *the new quotient is a whole number;* and so on.

Finally, adding the coefficient of x^{m-1}, to the preceding quotient, *the quotient of this sum by* a, *must be equal to* -1; hence, *the result of the addition of* 1, *which is the coefficient of* x^m, *to the preceding quotient, must be equal to* 0.

Any number that satisfies these conditions is a root, and any number that does not satisfy them is not a root.

All the entire roots may be determined by a single operation by the following

RULE.

I. Having determined all the divisors of the last term, write those that are comprehended between the limits $+L$ *and* $-L'$ *on the same horizontal line; underneath these write the quotients of the last term by each:*

II. Add the coefficient of x^1 *to each of these quotients, and write the sums underneath the corresponding quotients. Then divide these sums by each of the divisors, and write the quotients underneath the corresponding sums, rejecting fractional quotients and the divisors that produce them;* and so on.

When there are terms wanting in the proposed equation, their coefficients are to be regarded as equal to 0.

EXAMPLES.

1. Find the entire roots of the equation,

$$x^4 - x^3 - 13x^2 + 16x - 48 = 0.$$

A superior limit of the positive roots of this equation (Art. **279**), is $13 + 1 = 14$. The coefficient 48 need not be considered, since the last two terms can be put under the form $16(x - 3)$; hence, when $x > 3$, this part is essentially positive.

A superior limit of the negative roots (Art. **280**), is

$$-(1 + \sqrt{48}), \quad \text{or} \quad -8.$$

Therefore, the divisors of the last term that may be roots, are 1, 2, 3, 4, 6, 8, 12; moreover, neither $+1$, nor -1, will satisfy the equation, because the coefficient -48 is itself greater than the sum of all the others: we should therefore try only the *positive divisors* from 2 to 12, and the *negative divisors* from -2 to -6 inclusively.

By observing the rule given above, we have

12,	8,	6,	4,	3,	2,	− 2,	− 3,	− 4,	− 6
− 4,	− 6,	− 8,	− 12,	− 16,	− 24,	+ 24,	+ 16,	+ 12,	+ 8
+ 12,	+ 10,	+ 8,	+ 4,	0,	− 8,	+ 40,	+ 32,	+ 28,	+ 24
+ 1,	..,	..,	+ 1,	0,	− 4,	− 20,	..,	− 7,	− 4
− 12,	..,	..,	− 12,	− 13,	− 17,	− 33,	..,	− 20,	− 17
− 1,	..,	..,	− 3,	..,	..,	..,	..,	+ 5,	
− 2,	..,	..,	− 4,	..,	..,	..,	..,	+ 4,	..
..,	..,	..,	− 1,	..,	..,	..,	..,	− 1,	..

The *first* line contains the divisors, and the *second* the quotients arising from the division of -48, by each divisor. The *third* line contains these quotients, each augmented by the coefficient $+16$; and the *fourth*, the quotients of these sums by each divisor; this excludes the divisors $+8$, $+6$, and -3.

The *fifth* contains the preceding quotients, each augmented by the coefficient -13, and the *sixth* contains the quotients of these sums by each divisor; the third condition excludes the divisors 3, 2, -2, and -6.

Finally, the *seventh* is the third line of quotients, each augmented by the coefficient -1, and the *eighth* contains the quotients of these sums by each divisor. The divisors $+4$ and -4 are the only ones that give -1; hence, $+4$ and -4 are the only entire roots of the equation.

In fact, if we divide

$$x^4 - x^3 - 13x^2 + 16x - 48,$$

by the product $(x-4)(x+4)$, or $x^2 - 16$, the quotient will be $x^2 - x + 3$, which placed equal to zero, gives

$$x = \frac{1}{2} \pm \frac{1}{2}\sqrt{-11};$$

therefore, the four roots are

$$4, \quad -4, \quad \frac{1}{2} + \frac{1}{2}\sqrt{-11} \quad \text{and} \quad \frac{1}{2} - \frac{1}{2}\sqrt{-11}.$$

2. Find the entire roots of the equation

$$x^4 - 5x^3 + 25x - 21 = 0.$$

3. Find the entire roots of the equation

$$15x^5 - 19x^4 + 6x^3 + 15x^2 - 19x + 6 = 0.$$

4. Find the entire roots of the equation

$$9x^6 + 30x^5 + 22x^4 + 10x^3 + 17x^2 - 20x + 4 = 0.$$

IV. STURM'S THEOREM.

286. A **Theorem** is the statement of a truth that requires demonstration.

Sturm's Theorem.—The object of this theorem is to nd the number and places of the real roots of a numerical ation.

287. The theorem supposes that the equation contains no equal roots, which supposition may always be made, since we have the means of freeing an equation from its equal roots (Art. **271**). Let

$$X = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

be such an equation, and let X_1 denote the first derived polynomial of X.

Apply to X and X_1 a process similar to that for finding their greatest common divisor, but differing from it in this respect: that we *change the sign of each remainder* as soon as found; and furthermore, we avoid *introducing or rejecting any factors except positive ones.*

Since the equation $X = 0$ has no equal roots, there can be no common divisor of X and X_1, and consequently, the last remainder will not be equal to 0, but it will be independent of x.

If we denote the modified remainders by $X_2, X_3 \ldots X_r$ (which are read X *second,* X *third,* &c.), and the corresponding quotients by $Q_1, Q_2 \ldots Q_{r-1}$, we may write the following equations:

$$X = X_1Q_1 - X_2 \quad . \; . \; . \; . \; . \quad (2)$$

$$\left.\begin{aligned} X_1 &= X_2Q_2 - X_3 \\ &. \; . \; . \; . \; . \; . \; . \; . \; . \; . \\ X_{n-1} &= X_nQ_n - X_{n+1} \\ &. \; . \; . \; . \; . \; . \; . \; . \; . \; . \\ X_{r-2} &= X_{r-1}Q_{r-1} - X_r \end{aligned}\right\} \quad . \; . \; . \; . \quad (3)$$

In these equations, the quantities X, X_1, &c., represent the dividends and divisors actually used, that is, including the *positive factors* and *divisors* introduced. This modification does not affect their relative signs.

288. Theorem.—If a number p be substituted for x, in each of the expressions,

$$X, \; X_1, \; X_2, \; X_3 \ldots X_{r-1},$$

and the signs of the results, together with the sign of X_r, be arranged in a line; and if a second number q, algebraically greater than p, be substituted for x, and the signs arranged in like manner, then will *the number of variations of signs in the first line*, diminished by *the number of variations of signs in the second line*, be *the number of real roots lying between p and q.*

289. The demonstration of this theorem depends on the following properties of X, X_1, $X_2 \ldots X_n \ldots$ &c.:

1°. *Of these expressions no two consecutive ones can reduce to* 0 *for the same value of x.*

For, let X_{n-1}, X_n, X_{n+1}, be any three consecutive expressions. Then among equations (3), we shall find

$$X_{n-1} = X_n Q_n - X_{n+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

from which it appears that, if X_{n-1} and X_n should both become 0 for a value of x, X_{n+1} would be 0 for the same value; and since the equation which follows (4) must be

$$X_n = X_{n+1} Q_{n+1} - X_{n+2},$$

we shall have $X_{n+2} = 0$ for the same value, and so on until we should find $X_r = 0$, which cannot be; hence, X_{n-1} and X_n cannot both become 0 for the same value of x.

2°. *If any one of these expressions reduces to* 0 *for a particular value of x, the preceding and following ones will have contrary signs for that value.*

Suppose, for example, that X_n reduces to 0 for a particular value of x; an examination of equation (4), shows that X_{n-1} and X_{n+1}, will have contrary signs for the same value, inasmuch as one becomes equal to *minus* the other.

3°. *If a number insensibly less than one of the real ots of $X = 0$ be substituted for x in X and X_1, the* *results will have contrary signs; and if a number insensibly greater than this root be substituted, the results will have the same sign.*

Let us substitute $a + u$ for x in the expressions X and X_1, and designate by U and U_1 what they respectively become under this supposition. Then (Art. **267**), we have

$$\left.\begin{aligned} U &= A + A'u + A''\frac{u^2}{2} + \quad \&c. \\ U_1 &= A_1 + A'_1u + A''_1\frac{u^2}{2} + \quad \&c. \end{aligned}\right\} \quad . \; . \; . \; . \; (5)$$

in which A, A′, A″, &c., are the results obtained by the substitution of a for x, in X and its derived polynomials; and A_1, A'_1, &c., are similar results derived from X_1. If, now, a be a root of the proposed equation $X = 0$, then $A = 0$, and since A' and A_1 are each derived from X_1, by the substitution of a for x, we have $A' = A_1$, and equations (5) become

$$\left.\begin{aligned} U &= A'u + A''\frac{u^2}{2} + \quad \&c. \\ U_1 &= A' \;\; + A'_1u \;\; + \quad \&c. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; (6)$$

Now, the arbitrary quantity u may be taken so small that the signs of the values of U and U_1 will depend on the signs of their first terms (Art. **278**); that is, they will be alike when u is positive, or when $a + u$ is substituted for x, and unlike when u is negative, or when $a - u$ is substituted for x, *which was to be shown.*

290. Let us now see what changes take place in the number of *permanences* and *variations* of the signs of the quantities

$$X, \quad X_1, \quad X_2, \; . \; . \; . \; X_r.$$

when x is made to increase according to the law of continuity from the value p to the value q. As x increases, suppose it to pass in succession over values that reduce some of the quantities X, X_1, &c., to 0; from property 1°, no single value of x will be met with that reduces two consecutive ones to 0.

There may be two cases: 1°, a value of x may be passed that reduces X to 0; or, 2°, a value of x may be passed that reduces some of the other expressions to 0.

1°. Suppose, $x = a$, reduces X to 0; then, a is a real root of the given equation, and from property 3°, it follows, that just before reaching a, the corresponding values of X and X_1 have *contrary* signs, that is, in passing from the former to the latter there is a *variation* of sign; but from the same principle it follows that just after x passes over the value a, the corresponding values of X and X_1 have the *same* sign, that is, in passing from the former to the latter there is a *permanence.* Hence, when the value of x passes over the root a, one *variation* is lost. In like manner it may be shown that one variation is lost whenever the value of x passes over any other real root. Hence, the total number of variations *lost* while x is varying from p to q, is equal to the *number* of the real roots of the given equation lying between those numbers.

2°. Suppose, $x = m$, reduces some one of the other expressions, as X_n, to 0; then, from properties 1° and 2° it follows that neither X_{n-1} nor X_{n+1} can reduce to 0, for $x = m$, but must have contrary signs; hence, in passing from X_{n-1} to X_{n+1} (omitting X_n) we have *one variation* and *only one.* But both X_{n-1} and X_{n+1} are finite, and consequently neither can change its sign without passing through 0; therefore, their relative signs just before x reaches the value m and just after it passes that value, are
me, that is, the sign of one of them must be *the same*
the sign of the other must be *contrary to,* that of
onding values of X_n; hence, in passing over the

three, either just before X_n becomes 0 or just after, there is *one* and *only one* variation. Therefore, the reduction of X_n to 0 neither increases nor diminishes the number of variations: and this will evidently be the case, if several of the expressions X_1, X_2, &c., become 0 at the same time.

From what precedes we conclude that, *if we substitute p for x in the expressions*

$$X, \ X_1, \ X_2, \ldots X_{r-1}$$

and write down the signs, together with the sign of X_r, *in a line, and again substitute q for x and write down the signs as before, then will the number of variations of signs in the first line, diminished by the number of variations in the second line, be equal to the number of real roots of the equation lying between p and q;* which was to be proved.

If p is an inferior limit of the real roots, and q a superior limit, then will the rule give the whole number of real roots of the equation. It will be observed that the variation is lost when x passes from the preceding state to the root, and *not* when x passes from the root to the succeeding state. Hence, if p is a root of the given equation it is not to be counted amongst those given by the rule, but if q is a root of the equation it will form one of those given by the rule.

291. If in the application of the preceding principles, we find that any one of the expressions X_1, X_2 ... &c., X_n for instance, will preserve the same sign for all values of x in passing from p to q, inclusively, it will be unnecessary to use the succeeding expressions, or even to deduce them. For, as X_n preserves the same sign during the successive substitutions, it is plain that the same number of variations will be lost among the expressions X, X_1, &c. ... ending with X_n as among all including X_r. Whenever then, in the course of the division, it is found that by placing any of the remainders equal to 0, an equation is obtained with imaginary roots only (Art. **283**), it will be useless to obtain any of the succeeding remainders. This principle will be found useful in the solution of numerical examples.

292. As all the real roots of the proposed equation are necessarily included between $-\infty$ and $+\infty$, we may, by ascertaining the number of variations lost by the substitution of these, in succession, in the expressions $X, X_1 \dots X_n, \dots$ &c., readily determine the total number of such roots. It should be observed, that it will be only necessary to make these substitutions in the first terms of each of the expressions, as in this case the sign of the term will determine that of the entire expression (Art. **278**).

Having found the number of real roots, if we subtract this number from the greatest exponent of the unknown quantity, the remainder will be the number of imaginary roots (Art. **256**).

293. Having thus obtained the total number of real roots, we may ascertain their places by substituting for x, in succession, the values 0, 1, 2, 3, &c., until we find an entire number which gives the same number of variations as $+\infty$. This will be the smallest superior limit of the roots in whole numbers.

Then substitute -1, -2, &c., until a negative number is obtained which gives the same number of variations as $-\infty$. This will be the greatest inferior limit of the roots in whole numbers. Now, by commencing with this limit and observing the number of variations lost in passing from each number to the next in order, we shall discover how many roots are included between each two of the consecutive numbers used, and thus, of course, know the entire part of each root. The decimal part may then be sought by some of the known methods of approximation.

EXAMPLES.

1. Let $$8x^3 - 6x - 1 = 0 = X.$$

The first derived polynomial (Art. **267**) is

$$24x^2 - 6,$$

and since we may omit the positive factor 6, without affecting the sign, we may write

$$4x^2 - 1 = X_1.$$

Dividing X by X_1, we obtain for the first remainder, $-4x - 1$. Changing its sign, we have

$$4x + 1 = X_2.$$

Multiplying X_1 by the positive number 4, and then dividing by X_2, we obtain the second remainder -3; and by changing its sign

$$+3 = X_3.$$

The expressions to be used are then

$$X = 8x^3 - 6x - 1, \quad X_1 = 4x^2 - 1, \quad X_2 = 4x + 1, \quad X_3 = +3.$$

Substituting $-\infty$ and then $+\infty$, we obtain the two following arrangements of signs:

$- + - + \ldots\ldots 3$ *variations,*

$+ + + + \ldots\ldots 0$ "

there are then *three* real roots.

If, now, in the same expressions we substitute 0 and +1, and then 0 and −1, for x, we shall obtain the three following results:

For	$x = +1$	$+ + + +$	0 *variations.*
"	$x = 0$	$- - + +$	1 "
"	$x = -1$	$- + - +$	3 "

As $x = +1$ gives the same number of variations as $+\infty$, and $x = -1$ gives the same as $-\infty$, $+1$ and -1 are the superior and inferior limits in whole numbers. In passing from -1 to 0, *two* variations are lost, and in passing from 0 to $+1$, *one* variation is lost; hence, there are two negative roots between -1 and 0, and one positive root between 0 and $+1$.

2. Let $2x^4 - 13x^2 + 10x - 19 = 0$.

If we deduce X_1, and X_2, we have the expressions,

$$X = 2x^4 - 13x^2 + 10x - 19,$$

$$X_1 = 4x^3 - 13x + 5,$$

$$X_2 = 13x^2 - 15x + 38.$$

If we place $X_2 = 0$, we find that both the roots of the resulting equation are imaginary; hence, X_2 will be positive for all values of x (Art. **283**). It is then useless to seek for X_3 and X_4.

By the substitution of $-\infty$ and $+\infty$ in X, X_1, and X_2, we obtain for the first, *two* variations, and for the second, *none;* hence, there are two real and two imaginary roots in the proposed equation.

Find the number of real roots of the following equation:

3. $x^3 - 5x^2 + 8x - 1 = 0$.

Ans. 1 real root.

4. Find the number, places, and limits of the real roots of the equation,

$$x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$$

Ans. There are 4 real roots; one between 0 and 1, one between 2 and 3, one between 5 and 6, and one between -1 and 0. The limits are -1 and 6.

5. Find the number, places, and limits of the real roots of the equation, $x^3 - 23x - 24 = 0$.

Ans. 3 real roots, one between 5 and 6, one between -1 and -2, and one between -4 and -5. The limits are, -5 and $+6$.

6. Find the number, places, and limits of the real roots of the equation, $x^3 + \frac{3}{2}x^2 - 2x - 5 = 0$.

Ans. There is 1 real root, and it lies between the limits 1 and 2.

294. Cubic Equation.—In the preceding discussion we have supposed the equations to be given, and from relations existing between the coefficients of the different powers of the unknown quantity, have determined the number and places of the real roots; and, consequently, the number of imaginary roots.

In the equation of the second degree, we pointed out the relations that exist between the coefficients of the different powers of the unknown quantity when the roots are real, and when they are imaginary (Art. **125**).

Let us determine corresponding relations between the coefficients of an equation of the third degree.

Take the equation,

$$x^3 + Px^2 + Qx + U = 0,$$

causing the second term to disappear (Art. **266**), it will take the form,

$$x^3 + px + q = 0.$$

Hence, we have,

$$\begin{aligned} X &= x^3 + px + q, \\ X_1 &= 3x^2 + p, \\ X_2 &= -2px - 3q, \\ X_3 &= -4p^3 - 27q^2. \end{aligned}$$

In order that all the roots be real, the substitution of ∞ for x in the above expressions must give three permanences; and the substitution of $-\infty$ for x must give three variations. But the first supposition can only give three permanences when

$$-4p^3 - 27q^2 > 0;$$

a condition that requires p to be negative.

If, then, p be negative, we have, for $x = \infty$,

$$-4p^3 - 27q^2 > 0; \text{ that is, positive:}$$

or, $$4p^3 + 27q^2 < 0; \text{ that is, negative:}$$

hence, $\frac{q^2}{4} + \frac{p^3}{27} < 0$, which requires p to be negative, and $\frac{p^3}{27} > \frac{q^2}{4}$; conditions which indicate that the roots are all real.

V. CUBIC EQUATIONS.

295. Cardan's Rule.

Let, $$x^3 + px + q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be a cubic equation after being freed of its second term.

Take $$x = y + z;$$

then $$x^3 = y^3 + z^3 + 3yz(y + z);$$

transposing, and substituting x for $y + z$, we have,

$$x^3 - 3yz \times x - (y^3 + z^3) = 0 \quad . \quad . \quad . \quad . \quad (2)$$

Comparing this with equation (1), we have,

$$-3yz = p; \text{ and } y^3 + z^3 = -q \quad . \quad . \quad . \quad . \quad (3)$$

From the first of equations (3), we have,

$$z^3 = -\frac{p^3}{27y^3},$$

which, being substituted in the second, gives

$$y^3 - \frac{p^3}{27y^3} = -q;$$

clearing of fractions, and reducing

$$y^6 + qy^3 = \frac{p^3}{27}.$$

Solving this trinomial equation (Art. **132**), we have,

$$y^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}},$$

and the corresponding value of z^3, gives

$$z^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

But since $x = y + z$, we have,

$$x = \sqrt[3]{\left[-\frac{q}{2} + \sqrt{\left(\frac{q^2}{4} + \frac{p^3}{27}\right)}\right]} + \sqrt[3]{\left[-\frac{q}{2} - \sqrt{\left(\frac{q^2}{4} + \frac{p^3}{27}\right)}\right]}.$$

This is called *Cardan's Formula.*

By examining the above formula, it will be seen, that it is inapplicable to the case, when the quantity

$$\frac{q^2}{4} + \frac{p^3}{27},$$

under the radical sign of the second degree is negative; hence, it is only applicable to the case in which two of the roots are imaginary (Art. **294**).

Having found the real root, divide out the corresponding binomial factor and there will result an equation of the second degree that may be solved.

EXAMPLES.

1. What are the roots of the equation

$$x^3 - 6x^2 + 10x = 8?$$

Ans. $4, \quad 1 + \sqrt{-1}, \quad 1 - \sqrt{-1}.$

2. What are the roots of the equation

$$x^3 - 9x^2 + 28x = 30?$$

Ans. $3, \quad 3 + \sqrt{-1}, \quad 3 - \sqrt{-1}.$

3. What are the roots of the equation

$$x^3 - 7x^2 + 14x = 20?$$

Ans. $5, \quad 1 + \sqrt{-3}, \quad 1 - \sqrt{-3}.$

VI. HORNER'S METHOD OF SOLVING HIGHER EQUATIONS.

296. The method of approximating to the roots of a numerical equation of any degree, discovered by the English mathematician, W. G. Horner, is a process of remarkable simplicity and elegance.

The process consists in a succession of transformations of one equation to another, each transformed equation having its roots less than those of the given equation by the difference between the true value of the roots of the given equation, and the part of the root expressed by the figures already found. Such figures of the root are called the *initial figures.* Let

$$V = x^m + Px^{m-1} + Qx^{m-2} \dots + Tx + U = 0. \quad (1)$$

be any equation, and suppose we have found a part of one of the roots, which we denote by m, and denote the remaining part of the root by r.

Let us now transform the given equation into another, whose roots are less by m, and we have (Art. **264**),

$$V' = r^m + P'r^{m-1} + Q'r^{m-2} \dots + T'r + U' = 0 \,.. (2)$$

Now, when r is a very small fraction, all the terms of the second member, except the last two, may be neglected, and the first figure, in the value of r, may be found from the equation

$$T'r + U' = 0; \quad \text{giving} \quad r = -\frac{U'}{T'}; \quad \text{that is,}$$

The first figure of r is equal to the absolute term of the transformed equation, divided by the penultimate coefficient.

If we transform equation (2) into another, whose roots are less than those of the previous equation by the first figure of r, and designate the remaining part by s, we have,

$$V'' = s^m + P''s^{m-1} + Q''s^{m-2} \dots + T''s + U'' = 0,$$

the roots of which will be less than those of the given equation by $m +$ the first figure of r. The first figure in the value of S is found from the equation,

$$T''s + U'' = 0, \quad \text{giving} \quad s = -\frac{U''}{T''}.$$

We may thus continue the transformations at pleasure, and each one will evolve a new figure of the root. Hence, to find the roots of numerical equations, we have the following

RULE.

I. Find the number and places of the real roots by Sturm's theorem, and set the negative roots aside:

II. Transform the given equation into another whose roots shall be less than those of the given *equation, by the initial figure or figures already found: then, by Sturm's theorem, find the places of the roots of this new equation, and the first figure of each will be the first decimal place in each of the required roots:*

III. Divide the absolute term of the transformed equation by the penultimate coefficient, which is called the trial divisor, and the first figure of the quotient will be the next figure of the root:

IV. Transform the last equation into another whose roots shall be less than those of the previous equation by the figure last found, and proceed in a similar manner until the root be found to the required degree of accuracy.

This method is one of approximation, and it may happen that the rejection of the terms preceding the penultimate term will affect the quotient figure of the root. To avoid this error, find the first decimal places of the root, also, by the theorem of Sturm, as in example 4, page 404, and when the results coincide for two consecutive places of decimals, those subsequently obtained by the divisors may be relied on.

When the decimal portion of a negative root is to be found, first transform the given equation into another by changing the signs of the alternate terms (Art. **283**), and then find the decimal part of the corresponding positive root of this new equation.

When several decimal places are found in the root, the operation may be shortened according to the method of contractions indicated in the examples.

297. Let us now work one example in full. Let us take the equation of the third degree,

$$x^3 - 7x + 7 = 0.$$

By Sturm's rule, we have the quantities (Art. **290**),

$$\begin{aligned} X &= x^3 - 7x + 7 \\ X_1 &= 3x^2 - 7 \\ X_2 &= 2x - 3 \\ X_3 &= +1. \end{aligned}$$

Hence, for $x = \infty$, we have $+ + + +$ no variation,
$x = -\infty$ " $- + - +$ three variations;

therefore, the equation has *three* real roots, two positive and one negative.

To determine the initial figures of these roots, we have

$$\begin{array}{lll} \text{for } x = 0 \ldots + - - + & \quad & \text{for } x = 0 \ldots + - - + \\ \phantom{\text{for }} x = 1 \ldots + - - + & & \phantom{\text{for }} x = -1 \ldots + - - + \\ \phantom{\text{for }} x = 2 \ldots + + + + & & \phantom{\text{for }} x = -2 \ldots + + - + \\ & & \phantom{\text{for }} x = -3 \ldots + + - + \\ & & \phantom{\text{for }} x = -4 \ldots - + - + \end{array}$$

hence, there are two roots between 1 and 2, and one between -3 and -4.

In order to ascertain the first figures in the decimal parts of the two roots situated between 1 and 2, we shall transform the preceding expressions into others, in which the value of x is diminished by 1. Thus, for the quantity X, we have this operation:

$$\begin{array}{l} 1 + 0 - 7 + 7\,(1 \\ \quad\ \underline{1 + 1 - 6} \\ \quad\ 1 - 6 + 1 \\ \quad\ \underline{1 + 2} \\ \quad\ 2 - 4 \\ \quad\ \underline{1} \\ \quad\ 3 \end{array}$$

And transforming the others in the same way, we obtain the quantities,

$$\begin{aligned} Y &= y^3 + 3y^2 - 4y + 1; \\ Y_1 &= 3y^2 + 6y - 4; \\ Y_2 &= 2y - 1; \\ Y_3 &= +1. \end{aligned}$$

Let	$y = .1$	we have	$+ - - +$	two variations,
	$y = .2$	"	$+ - - +$	"
	$y = .3$	"	$+ - - +$	"
	$y = .4$	"	$- - - +$	one variation,
	$y = .5$	"	$- - \mp +$	"
	$y = .6$	"	$- + + +$	"
	$y = .7$	"	$+ + + +$	no variation.

Therefore the initial figures of the two positive roots are 1.3, 1.6.

Let us now find the decimal part of the first root.

1.3 + 0	− 7	+ 7 (1.356895867
1	1	− 6
1	− 6	*1
1	2	− .903
2	*− 4	**.097
1	.99	− .086625
*3.3	− 3.01	***.010375
3	1.08	− .009048984
3.6	**− 1.93	****.001326016
3	.1975	− .001184430
**3.9 5	− 1.7325	.000141586
5	.2000	− .000132923
4.0 0	***− 1.5325	.000008663
5	.024336	− .000007382
***4.0 56	− 1.508164	.000001281
6	.024372	− .000001181
4.0 62	****− 1.48379|2	.000000100
6	.00325|4	− .000000089
****4.0|68 8	− 1.48053|8	.000000011
8	.00325|4	− .000000010
|4.0| 696	− 1.4772|8	1
	.0003|6	
	− 1.4769|2	
	.0003|6	
	− 1 | 4|4|7|6|5	

The operations in the example are performed as follows:

1st. We find the places and the initial figures of the positive roots, to include the first decimal place by Sturm's theorem.

2d. Then to find the decimal part of the first positive root, we arrange the coefficients, and perform a succession of transformations by Synthetical Division, which must begin with the initial figures already known.

We first transform the given equation into another whose roots shall be less by 1. The coefficients of this new equation are, 1, 3, —4, and 1, and are all, except the first, marked by a star. The root of this transformed equation, corresponding to the root sought of the given equation, is a decimal fraction of which we know the first figure 3.

We next transform the last equation into another whose roots are less by three-tenths, and the coefficients of the new equation are each marked by two stars.

The process here changes, and we find the next figure of the root by dividing the absolute term .097 by the penultimate coefficient — 1.93, giving .05 for the next figure of the root.

We again transform the equation into another whose roots shall be less by .05, and the coefficients of the new equation are marked by three stars.

We then divide the absolute term, .010375 by the penultimate coefficient, — 1.5325, and obtain .006, the next figure of the root: and so on for other figures.

In regard to the contractions, we may observe that, having decided on the number of decimal places to which the figures in the root are to be carried, we need not take notice of figures which fall to the right of that number in any of the dividends. In the example under consideration, we propose to carry the operations to the 9th decimal place of the root; hence, we may reject all the decimal places of the dividends after the 9th.

The fourth dividend, marked by four stars, contains nine decimal places, and the next dividend is to contain no more.

But the corresponding quotient figure 8, is the fourth figure from the decimal point; hence, at this stage of the operation, all the places of the divisor, after the 5th, may be omitted, since the 5th, multiplied by the 4th, will give the 9th order of decimals. Again: since each new figure of the root is removed one place to the right, one additional figure, in each subsequent divisor, may be omitted. The contractions, therefore, begin by striking off the two in the 4th divisor.

In passing from the first column to the second, in the next operation, we multiply by .0008; but since the product is to be limited to five decimal places, we need take notice of but one decimal place in the first column; that is, in the first operation of contraction, we strike off, in the first column, the two figures 68: and, generally, *for each figure omitted in the second column, we omit two in the first.*

It should be observed, that when places are omitted in either column, whatever would have been carried to the last figure retained, had no figures been omitted, is always to be added to that figure. Having found the figure 8 of the root, we need not annex it in the first column, nor need we annex any subsequent figures of the root, since they would all fall at the right, among the rejected figures. Hence, neither 8, nor any subsequent figures of the root, will change the available part of the first column.

In the next operation, we divide .000141586 by 1.4772, omitting the figure 8 of the divisor: this gives the figure 9 of the root. We then strike off the figures 4.0, in the first column, and multiplying by .00009, we form the next divisor in the second column, —1.4769, and the next dividend in the 3d column, .000008663. Striking off 5 in this divisor, we find the next figure of the root, which is 5.

It is now evident that the products from the first column, will fall in the second, among the rejected figures at the right; we need, therefore, in future, take no notice of them.

Omitting the right hand figure, the next divisor will be 1.476, and the next figure of the root 8. Then omitting 6 in the divisor, we obtain the quotient figure 8: omitting 7 we obtain 6, and omitting 4 we obtain 7, the last figure to be found. We have thus found the root $x = 1.356895876 \ldots$; and all similar examples are wrought after the same manner.

The next operation is to find the root whose initial figures are 1.6, to nine decimal places. The operations are entirely similar to those just explained.

We find for the second root, $x = 1.69202141$.

For the negative root, change the signs of the second and fourth terms (Art. **278**), and we have,

1 − 0	− 7	− 7 (3.0489173396
3	9	+ 6
3	2	− 1
3	18	.814464
6	20	− .185536
3	.3616	.166382592
9.04	20.3616	− 19153408
4	.3632	18791228
9.08	20.7248	−362180
4	73024	208875
9.128	20.797824	−153305
8	73088	146212
9.136	20.870912	−7093
8	8230	6266
. . 9.1 44	20.879142	−827
	8230	626
	20.88737	−201
	9	188
	20.88746	−13
	9	12
	20.8875	1

4. Find the roots of the equation

$$x^3 + 11x^2 - 102x + 181 = 0.$$

The functions are

$$\begin{aligned} X &= x^3 + 11x^2 - 102x + 181 \\ X_1 &= 3x^2 + 22x - 102 \\ X_2 &= 122x - 393 \\ X_3 &= + ; \end{aligned}$$

and the signs of the leading terms are all $+$; hence, the substitution of $-\infty$ and $+\infty$ must give three real roots.

To discover the situation of the roots, we make the substitutions

$x = 0$	which gives	$+ - - +$	two variations,
$x = 1$	"	$+ - - +$	"
$x = 2$	"	$+ - - +$	"
$x = 3$	"	$+ - - +$	"
$x = 4$	"	$+ + + +$	no variation;

hence, the two positive roots are between 3 and 4, and we must therefore transform the several functions into others, in which x shall be diminished by 3. Thus we have (Art. **296**)

$$\begin{aligned} Y &= y^3 + 20y^2 - 9y + 1 \\ Y_1 &= 3y^2 + 40y - 9 \\ Y_2 &= 122y - 27 \\ Y_3 &= + \end{aligned}$$

Make the following substitutions in these functions, viz.:

$y = 0$	signs	$+ - - +$	two variations,
$y = .1$	"	$+ - - +$	"
$y = .2$	"	$+ - - +$	"
$y = .3$	"	$+ + + +$	no variation;

hence, the two positive roots are between 3.2 and 3.3, and we must again transform the last functions into others, in which y shall be diminished by .2. Effecting this transformation, we have,

$$\begin{aligned} Z &= z^3 + 20.6z^2 - .88z + .008 \\ Z_1 &= 3z^2 + 41.2z - .88 \\ Z_2 &= 122z - 2.6 \\ Z_3 &= +. \end{aligned}$$

Let $z = 0$ then signs are $+ - - +$ two variations,
$z = .01$ " " $+ - - +$ "
$z = .02$ " " $- - - +$ one variation,
$z = .03$ " " $+ + + +$ no variation;

hence we have 3.21 and 3.22 for the positive roots, and the sum of the roots is -11; therefore, $-11 - 3.21 - 3.22 = -17.4$, is the negative root, nearly.

For the positive root, whose initial figures are 3.21, we have,

$$x = 3.21312775;$$

and for the root whose initial figures are 3.22, we have,

$$x = 3.229522121;$$

and for the negative root,

$$x = -17.44264896.$$

EXAMPLES.

1. Find a root of the equation $x^3 + x^2 + x - 100 = 0$.
Ans. 4.2644299731.

2. Find the roots of the equation $x^4 - 12x^2 + 12x - 3 = 0$.

$$Ans. \left\{\begin{array}{l} +\ 2.858083308163 \\ +\ .606018306917 \\ +\ .443276939605 \\ -\ 3.907378554685. \end{array}\right.$$

3. Find the roots of the equation

$$x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$$

$$\textit{Ans.} \left\{ \begin{array}{ll} + & 5.2360679775 \\ + & .7639320225 \\ + & 2.7320508075 \\ - & .7320508075. \end{array} \right.$$

4. Find the roots of the equation

$$x^5 - 10x^3 + 6x + 1 = 0.$$

$$\textit{Ans.} \left\{ \begin{array}{ll} - & 3.0653157912983 \\ - & .6915762804900 \\ - & .1756747992883 \\ + & .8795087084144 \\ + & 3.0530581626622. \end{array} \right.$$

THE NEW YORK PUBLIC LIBRARY
REFERENCE DEPARTMENT

This book is under no circumstances to be taken from the Building

form 410